

**ADVANCED ECONOMETRIC THEORY**  
**EXERCISES 15**  
**GENERAL ASYMPTOTIC TESTS**

1. Let  $\hat{\theta}_n$  be an estimator obtained by maximizing an objective function  $L_n(\theta)$ , where  $\theta \in \Theta \subseteq \mathbb{R}^p$ , and  $\theta_0 \in \Theta$  is a value of  $\theta$  such that

$$n^{-1/2} \frac{\partial L_n}{\partial \theta}(\theta_0) \xrightarrow[n \rightarrow \infty]{d} N[0, I_0]$$

and

$$-\frac{1}{n} \frac{\partial^2 L_n}{\partial \theta \partial \theta'}(\theta_0) \xrightarrow[n \rightarrow \infty]{} J_0 \text{ with probability 1}$$

where  $I_0$  and  $J_0$  are deterministic positive definite matrices. We wish to test the mixed hypothesis

$$H_0 : \{ \theta \mid \exists a \in A \subseteq \mathbb{R}^q : g(\theta, a) = 0 \}$$

where  $g$  is a differentiable function taking its values in  $\mathbb{R}^r$ , such that

$$\text{rank} \left[ \frac{\partial g}{\partial \theta'} \right] = r, \text{rank} \left[ \frac{\partial g}{\partial a'} \right] = q.$$

- (a) Describe the following criteria for testing  $H_0$  :
- (1) the Wald statistic,
  - (2) the Lagrange multiplier statistic,
  - (3) the score statistic,
  - (4) the “likelihood ratio” statistic.
- (b) Under usual regularity conditions, what are the asymptotic distributions of these criteria?
- (c) What form takes the Wald statistic in the case of an implicit constraint (without an auxiliary parameter  $a$ )? Which important execution difference do you observe?

- (d) Which important condition must be satisfied for a “likelihood ratio” statistic to follow a  $\chi^2$  distribution?
  - (e) What are the forms of the different statistics when  $I_0 = J_0$ ?
  - (f) Provide a condition under which the Lagrange multiplier and score statistics are identical.
2. Let  $h(Y_i, X_i, \theta_0), i = 1, \dots, n$ , be functions (taking their values in  $\mathbb{R}^H$ ) such that

$$E[h(Y_i, X_i, \theta_0)] = 0, i = 1, \dots, n,$$

where the observations  $(Y_i, X_i), i = 1, \dots, n$  are i.i.d. If we estimate this system by the generalized method of moments, describe a specification test for this system.

Reference: Gouriéroux and Monfort (1995, Chapter 18).

## References

GOURIÉROUX, C., AND A. MONFORT (1995): *Statistics and Econometric Models, Volumes One and Two*. Cambridge University Press, Cambridge, U.K., Translated by Quang Vuong.