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Time series and financial econometrics

Sign-based tests for medians and independence

1. Let $X_1, \ldots, X_T$ be independent observations with continuous distributions. $X_1, \ldots, X_T$ may have not have finite mean and may not be identically distributed. Set

$$\begin{aligned} u(x) &= 1\,, \quad \text{if } x \geq 0 \\ &= 0\,, \quad \text{if } x < 0\,. \end{aligned} \tag{1}$$

   (a) Define the median $\mathrm{Med}(X_t)$ of the random variable X_t ($1 \leq t \leq T$).

   (b) Derive the joint distribution of the random variables $u(X_1), \ldots, u(X_T)$.

   (c) Describe a procedure for testing the hypothesis

$$H_0 : X_1, \ldots, X_T \text{ all have zero median.} \tag{2}$$

   (d) Describe a procedure for testing the hypothesis

$$H_0 : \mathrm{Med}(X_t) = m_0\,,\ t = 1, \ldots, T\,, \tag{3}$$

   where m_0 is an arbitrary constant (possibly non-zero).

   (e) Describe a procedure for testing the hypothesis

$$H_0 : \mathrm{Med}(X_t) = a + bt\,,\ t = 1, \ldots, T\,, \tag{4}$$

   where a and b are fixed constants.

2. Let $X_1, \ldots, X_T$ be independent observations with continuous distributions. $X_1, \ldots, X_T$ may have not have finite mean and may not be identically distributed. Let k be nonnegative integer ($1 \leq k \leq T-1$).

   (a) Describe a procedure for testing the hypothesis that $X_1, \ldots, X_T$ are independent against an alternative where

$$\mathrm{Med}(X_t X_{t+k}) > 0\,,\ t = 1, \ldots, T-k. \tag{5}$$

   (b) Describe a procedure for testing the hypothesis that $X_1, \ldots, X_T$ are independent against an alternative where

$$\mathrm{Med}(X_t X_{t+k}) < 0\,,\ t = 1, \ldots, T-k. \tag{6}$$

(c) Describe a procedure for testing the hypothesis that $X_1, \ldots, X_T$ are independent against an alternative where

$$\text{Med}(X_t X_{t+k}) \neq 0\,,\; t = 1, \ldots, T-k. \tag{7}$$

(d) For $k = 1$, propose an interpretation of the proposed procedures in terms of "runs"?

(e) Discuss the validity of the procedures proposed above if $X_1, \ldots, X_T$ have finite second moments, but variances increase as t increases.

[Reference: Dufour (1981).]

3. Let $X_1, \ldots, X_T$ be independent observations with continuous distributions symmetric with respect to zero. $X_1, \ldots, X_T$ may have not have finite means and may not be identically distributed.

(a) Let

$$R_t^+ = \sum_{t=1}^{T} u(|X_t| - |X_i|)\,,\; t = 1, \ldots, T\,. \tag{8}$$

where

$$\begin{aligned} u(x) \quad &= 1\,, \quad \text{if } x \geq 0 \\ &= 0\,, \quad \text{if } x < 0\,. \end{aligned} \tag{9}$$

Propose a procedure based on $\big(u(X_1), \ldots, u(X_T)\big)'$ and $\big(R_1^+, \ldots, R_T^+\big)'$ for testing the hypothesis

$$H_0 : X_1, \ldots, X_T \text{ have zero median.} \tag{10}$$

Discuss the advantages and disadvantages of the latter procedure over a procedure only based on $\big(u(X_1), \ldots, u(X_T)\big)'$.

(b) Propose a procedure based on $\big(u(X_1), \ldots, u(X_T)\big)'$ and $\big(R_1^+, \ldots, R_T^+\big)'$ for testing the hypothesis that $X_1, \ldots, X_T$ are independent against an alternative where

$$\text{Med}(X_t X_{t+k}) > 0\,,\; t = 1, \ldots, T-k. \tag{11}$$

[Reference: Dufour (1981).]

References

DUFOUR, J.-M. (1981): "Rank Tests for Serial Dependence," *Journal of Time Series Analysis*, 2, 117–128.