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1. Generating functions and spectral density

Generating functions constitute a convenient technique for representing and determining the autocovariance structure of a stationary process.

**Definition 1.1 Generating function.** Let \((a_k : k = 0, 1, 2, \ldots)\) and \((b_k : k = \ldots, -1, 0, 1, \ldots)\) two sequences of complex numbers. Let \(D(a) \subseteq \mathbb{C}\) the set of points \(z \in \mathbb{C}\) at which the series \(\sum_{k=0}^{\infty} a_k z^k\) converges, and \(D(b) \subseteq \mathbb{C}\) the set of points \(z\) for which where the series \(\sum_{k=-\infty}^{\infty} b_k z^k\) converges. Then the functions

\[
a(z) = \sum_{k=0}^{\infty} a_k z^k, z \in D(a) \tag{1.1}
\]

and

\[
b(z) = \sum_{k=-\infty}^{\infty} b_k z^k, z \in D(b) \tag{1.2}
\]

are called the generating functions of the sequences \(a_k\) and \(b_k\) respectively.
Proposition 1.1  CONVERGENCE ANNULUS OF A GENERATING FUNCTION.  Let \((a_k : k \in \mathbb{Z})\) be a sequence of complex numbers. Then the generating function

\[
a(z) = \sum_{k=\infty}^{\infty} a_k z^k
\]

(1.3)

converges for \(R_1 < |z| < R_2\) where

\[
R_1 = \limsup_{k \to \infty} |a_{-k}|^{1/k},
\]

(1.4)

\[
R_2 = 1/\left[\limsup_{k \to \infty} |a_k|^{1/k}\right],
\]

(1.5)

and diverges for \(|z| < R_1\) or \(|z| > R_2\). If \(R_2 < R_1\), \(a(z)\) converges nowhere and, if \(R_1 = R_2\), \(a(z)\) diverges everywhere except possibly, for \(|z| = R_1 = R_2\). Further, when \(R_1 < R_2\), the coefficients \(a_k\) are uniquely defined, and

\[
a_k = \frac{1}{2\pi i} \int_C \frac{a(z) \, dz}{(z - z_0)^{k+1}}, \quad k = 0, \pm 1, \pm 2, \ldots
\]

(1.6)

where \(C = \{z \in \mathbb{C} : |z - z_0| = R\}\) and \(R_1 < R < R_2\).
Proposition 1.2 Sums and Products of Generating Functions. Let \((a_k : k \in \mathbb{Z})\) and \((b_k \in \mathbb{Z})\) two sequences of complex numbers such that the generating functions \(a(z)\) and \(b(z)\) converge for \(R_1 < |z| < R_2\), where \(0 \leq R_1 < R_2 \leq \infty\). Then,

1. the generating function of the sum \(c_k = a_k + b_k\) is \(c(z) = a(z) + b(z)\);
2. if the product sequence

\[
d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j}
\]

(1.7)

converges for any \(k\), the generating function of the sequence \(d_k\) is

\[
d(z) = a(z)b(z).
\]

(1.8)

Further, the series \(c(z)\) and \(d(z)\) converge for \(R_1 < |z| < R_2\).

We will be especially interested by generating functions of autocovariances \(\gamma_k\) and autocorrelations \(\rho_k\) of a second-order stationary process \(X_t\):

\[
\gamma_x(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k,
\]

(1.9)

\[
\rho_x(z) = \sum_{k=-\infty}^{\infty} \rho_k z^k = \gamma_x(z)/\gamma_0.
\]

(1.10)

We see immediately that the generating function with a white noise \(\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)\) is constant:

\[
\gamma_u(z) = \sigma^2, \rho_u(z) = 1.
\]

(1.11)
Proposition 1.3  CONVERGENCE OF AUTOCOVARIANCE GENERATING FUNCTIONS.  Let $\gamma_k, k \in \mathbb{Z}$, the autocovariances of a second-order stationary process $X_t$, and $\rho_k, k \in \mathbb{Z}$, the corresponding autocorrelations.

1. If $R \equiv \limsup_{k \to \infty} |\rho_k|^{1/k} < 1$, the generating functions $\gamma_x(z)$ and $\rho_x(z)$ converge for $R < |z| < 1/R$.

2. If $R = 1$, the functions $\gamma_x(z)$ and $\rho_x(z)$ diverge everywhere, except possibly on the circle $|z| = 1$.

3. If $\sum_{k=0}^{\infty} |\rho_k| < \infty$, the functions $\gamma_x(z)$ and $\rho_x(z)$ converge absolutely and uniformly on the circle $|z| = 1$.

Proposition 1.4  IDENTIFIABILITY OF AUTOCOVARIANCES AND AUTOCORRELATIONS BY GENERATING FUNCTIONS.  Let $\gamma_k$ and $\rho_k, k \in \mathbb{Z}$, autocovariance and autocorrelation sequences such that

\begin{align}
\gamma(z) &= \sum_{k=-\infty}^{\infty} \gamma_k z^k = \sum_{k=-\infty}^{\infty} \gamma'_k z^k, \\
\rho(z) &= \sum_{k=-\infty}^{\infty} \rho_k z^k = \sum_{k=-\infty}^{\infty} \rho'_k z^k
\end{align}

(1.12) (1.13)

where the series considered converge for $R < |z| < 1/R$, where $R \geq 0$. Then $\gamma_k = \gamma'_k$ and $\rho_k = \rho'_k$ for any $k \in \mathbb{Z}$.
Proposition 1.5 Generating function of the autocovariances of a MA($\infty$) process. Let \( \{X_t : t \in \mathbb{Z}\} \) a second-order stationary process such that

\[
X_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}
\]  

(1.14)

where \( \{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \). If the series

\[
\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j
\]

(1.15)

and \( \psi(z^{-1}) \) converge absolutely, then

\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1}).
\]

(1.16)
Corollary 1.6 Generating function of the autocovariances of an ARMA process. Let \( \{X_t : t \in \mathbb{Z}\} \) a second-order stationary and causal ARMA\((p,q)\) process, such that

\[
\phi(B)X_t = \bar{\mu} + \theta(B)u_t
\]  

(1.17)

where \( \{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \), \( \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \) and \( \theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q \). Then the generating function of the autocovariances of \( X_t \) is

\[
\gamma_x(z) = \sigma^2 \frac{\theta(z) \theta(z^{-1})}{\phi(z) \phi(z^{-1})}
\]  

(1.18)

for \( R < |z| < 1/R \), where

\[
0 < R = \max\{|G_1|, |G_2|, \ldots, |G_p|\} < 1
\]  

(1.19)

and \( G_1^{-1}, G_2^{-1}, \ldots, G_p^{-1} \) are the roots of the polynomial \( \phi(z) \).
Proposition 1.7 Generating function of the autocovariances of a filtered process. Let \( \{X_t : t \in \mathbb{Z}\} \) a second-order stationary process and
\[
Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}, t \in \mathbb{Z},
\]
(1.20)
where \((c_j : j \in \mathbb{Z})\) is a sequence of real constants such that \(\sum_{j=-\infty}^{\infty} |c_j| < \infty\). If the series \(\gamma_x(z)\) and \(c(z) = \sum_{j=-\infty}^{\infty} c_j z^j\) converge absolutely, then
\[
\gamma_y(z) = c(z)c(z^{-1})\gamma_x(z).
\]
(1.21)
**Definition 1.2 Spectral Density.** Let $X_t$ a second-order stationary process such that the generating function of the autocovariances $\gamma_x(z)$ converge for $|z| = 1$. The spectral density of the process $X_t$ is the function

$$f_x(\omega) = \frac{1}{2\pi} \left[ \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\omega k) \right]$$

$$= \frac{\gamma_0}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma_k \cos(\omega k) \quad (1.22)$$

where the coefficients $\gamma_k$ are the autocovariances of the process $X_t$. The function $f_x(\omega)$ is defined for all the values of $\omega$ such that the series $\sum_{k=1}^{\infty} \gamma_k \cos(\omega k)$ converges.

**Remark 1.1** If the series $\sum_{k=1}^{\infty} \gamma_k \cos(\omega k)$ converges, it is immediate that $\gamma_x(e^{-i\omega})$ converge and

$$f_x(\omega) = \frac{1}{2\pi} \gamma_x(e^{-i\omega}) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\omega k} \quad (1.23)$$

where $i = \sqrt{-1}$. 
Proposition 1.8  CONVERGENCE AND PROPERTIES OF THE SPECTRAL DENSITY. Let $\gamma_k, k \in \mathbb{Z}$, be an autocovariance function such that $\sum_{k=0}^{\infty} |\gamma_k| < \infty$. Then

1. the series

$$f_x(\omega) = \gamma_0 + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma_k \cos(\omega k)$$

converges absolutely and uniformly in $\omega$;

2. the function $f_x(\omega)$ is continuous;

3. $f_x(\omega + 2\pi) = f_x(\omega)$ and $f_x(-\omega) = f_x(\omega)$, $\forall \omega$;

4. $\gamma_k = \int_{-\pi}^{\pi} f_x(\omega) \cos(\omega k) d\omega, \forall k$;

5. $f_x(\omega) \geq 0$;

6. $\gamma_0 = \int_{-\pi}^{\pi} f_x(\omega) d\omega$. 
Proposition 1.9  SPECTRAL DENSITIES OF SPECIAL PROCESSES. Let \( \{X_t : t \in \mathbb{Z}\} \) be a second-order stationary process with autocovariances \( \gamma_k, k \in \mathbb{Z} \).

1. If \( X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \) where \( \{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \) and \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \), then

\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \psi(e^{i\omega}) \psi(e^{-i\omega}) = \frac{\sigma^2}{2\pi} |\psi(e^{i\omega})|^2.
\] (1.25)

2. If \( \phi(B)X_t = \bar{\mu} + \theta(B)u_t \), where \( \phi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p \), \( \theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q \) and \( \{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \), then

\[
f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta(e^{i\omega})}{\phi(e^{i\omega})} \right|^2
\] (1.26)

3. If \( Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j} \) where \( \{c_j : j \in \mathbb{Z}\} \) is a sequence of real constants such that \( \sum_{j=-\infty}^{\infty} |c_j| < \infty \), and if \( \sum_{k=0}^{\infty} |\gamma_k| < \infty \), then

\[
f_y(\omega) = |c(e^{i\omega})|^2 f_x(\omega).
\] (1.27)
2. Inverse autocorrelations

**Definition 2.1** **Inverse Autocorrelations.** Let \( f_x(\omega) \) the spectral density of a second-order stationary process \( \{X_t : t \in \mathbb{Z}\} \). If the function \( 1/f_x(\omega) \) is also a spectral density, the autocovariances \( \gamma_x^{(I)}(k), k \in \mathbb{Z} \), associated with the inverse spectrum inverse \( 1/f_x(\omega) \) are called the inverse autocovariances of the process \( X_t \), i.e.

\[
\gamma_x^{(I)}(k) = \int_{-\pi}^{\pi} \frac{1}{f_x(\omega)} \cos(\omega k) d\omega, k \in \mathbb{Z}.
\] (2.1)

The inverse autocovariances satisfy the equation

\[
\frac{1}{f_x(\omega)} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_x^{(I)}(k) \cos(\omega k) = \frac{1}{2\pi} \gamma_x^{(I)}(0) + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma_x^{(I)} \cos(\omega k).
\] (2.2)

The inverse autocorrelations are

\[
\rho_x^{(I)}(k) = \frac{\gamma_x^{(I)}(k)}{\gamma_x^{(I)}(0)}, k \in \mathbb{Z}.
\] (2.3)

A sufficient condition for the function \( 1/f_x(\omega) \) to be a spectral density is that the function \( 1/f_x(\omega) \) be continuous on the interval \(-\pi \leq \omega \leq \pi\), which entails that \( f_x(\omega) > 0, \forall \omega \).
If the process $X_t$ is a second-order stationary $ARMA(p, q)$ process such that

$$
\phi_p(B)X_t = \bar{\mu} + \theta_q(B)u_t
$$

(2.4)

where $\phi_p(B) = 1 - \phi_1B - \cdots - \phi_pB^p$ and $\theta_q(B) = 1 - \theta_1B - \cdots - \theta_qB^q$ are polynomials whose roots are all outside the unit circle and $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$, then

$$
f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q(e^{i\omega})}{\phi_p(e^{i\omega})} \right|^2
$$

(2.5)

$$
\frac{1}{f_x(\omega)} = \frac{2\pi}{\sigma^2} \left| \frac{\phi_p(e^{i\omega})}{\theta_q(e^{i\omega})} \right|^2
$$

(2.6)

The inverse autocovariances $\gamma_x^{(I)}(k)$ are the autocovariances associated with the model

$$
\theta_q(B)X_t = \bar{\mu} + \phi_p(B)v_t
$$

(2.7)

where $\{v_t : t \in \mathbb{Z}\} \sim WN(0, 1/\sigma^2)$ and $\bar{\mu}$ is some constant. Consequently, the inverse autocorrelations of an $ARMA(p, q)$ process behave like the autocorrelations of an $ARMA(q, p)$. For an process $AR(p)$ process,

$$
\rho_x^{(I)}(k) = 0, \text{ for } k > p.
$$

(2.8)

For a $MA(q)$ process, the inverse partial autocorrelations (i.e. the partial autocorrelations associated with the inverse autocorrelations) are equal to zero for $k > q$. These properties can be used for identifying the order of a process.
3. Multiplicity of representations

3.1. Backward representation ARMA models

By the backward Wold theorem, we know that any strictly indeterministic second-order stationary process \( X_t : t \in \mathbb{Z} \) can be written in the form

\[
X_t = \mu + \sum_{j=0}^{\infty} \bar{\psi}_j \bar{u}_{t+j}
\]

(3.1)

where \( \bar{u}_t \) is a white noise such that \( E(X_{t-j} \bar{u}_t) = 0 \), \( \forall j \geq 1 \). In particular, if

\[
\varphi_p(B)(X_t - \mu) = \theta_q(B)u_t
\]

(3.2)

where the polynomials \( \varphi_p(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p \) and \( \theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q \) have all their roots outside the unit circle and \( \{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \), the spectral density of \( X_t \) is

\[
f_X(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{\theta_q(e^{i\omega})}{\varphi_p(e^{i\omega})} \right|^2.
\]

(3.3)
Consider the process
\[ Y_t = \frac{\varphi_p(B^{-1})}{\theta_q(B^{-1})} (X_t - \mu) = \sum_{j=0}^{\infty} c_j (X_{t+j} - \mu). \quad (3.4) \]

By Proposition 1.9, the spectral density of \( Y_t \) is
\[ f_y(\omega) = \left| \frac{\varphi_p(e^{i\omega})}{\theta_q(e^{i\omega})} \right|^2 f_x(\omega) = \frac{\sigma^2}{2\pi} \quad (3.5) \]

and thus \( \{Y_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2) \). If we define \( \bar{u}_t = Y_t \), we see that
\[ \frac{\varphi_p(B^{-1})}{\theta_q(B^{-1})} (X_t - \mu) = \bar{u}_t \quad (3.6) \]
or
\[ \varphi_p(B^{-1})X_t = \bar{\mu} + \theta_q(B^{-1})\bar{u}_t, \quad (3.7) \]

and
\[ X_t - \varphi_1X_{t+1} - \cdots - \varphi_pX_{t+p} = \bar{\mu} + \bar{u}_t - \theta_1\bar{u}_{t+1} - \cdots - \theta_q\bar{u}_{t+q} \quad (3.8) \]

where \( (1 - \varphi_1 - \cdots - \varphi_p)\mu = \bar{\mu} \). We call (3.6) or (3.8) the backward representation of the \( X_t \) process.
3.2. Multiple moving-average representations

Let \( \{X_t\} \sim \text{ARIMA}(p, d, q) \). Then

\[
W_t = (1 - B)^d X_t \sim \text{ARMA}(p, q).
\] (3.9)

If we suppose that \( E(W_t) = 0 \), \( W_t \) satisfies an equation of the form

\[
\phi_p(B)W_t = \theta_q(B)u_t
\] (3.10)

or

\[
W_t = \frac{\theta_q(B)}{\phi_p(B)} u_t = \psi(B)u_t.
\] (3.11)

To determine an appropriate \( \text{ARMA} \) model, one typically estimates the autocorrelations \( \rho_k \). The latter are uniquely determined by the generating function of the autocovariances:

\[
\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1}) = \sigma^2 \frac{\theta_q(z)}{\phi_p(z)} \frac{\theta_q(z^{-1})}{\phi_p(z^{-1})}.
\] (3.12)

If

\[
\theta_q(z) = 1 - \theta_1 z - \cdots - \theta_q z^q = (1 - H_1 z) \cdots (1 - H_q z) = \prod_{j=1}^q (1 - H_j z),
\] (3.13)

then

\[
\gamma_x(z) = \frac{\sigma^2}{\phi_p(z) \phi_p(z^{-1})} \prod_{j=1}^q (1 - H_j z)(1 - H_j z^{-1}).
\] (3.14)
However

\[(1 - H_j z)(1 - H_j z^{-1}) = 1 - H_j z - H_j z^{-1} + H_j^2 = H_j^2 (1 - H_j z^{-1} - H_j z^{-1} + H_j^{-2})
\]
\[= H_j^2 (1 - H_j z)(1 - H_j z^{-1})
\]  

(3.15)

hence

\[
\gamma_x(z) = \left[ \sigma^2 \prod_{j=1}^{q} H_j^2 \right] \prod_{j=1}^{q} (1 - H_j z^{-1}) \left( 1 - H_j^{-1} z^{-1} \right) = \bar{\sigma}^2 \frac{\theta'_q(z) \theta'_q(z^{-1})}{\varphi_p(z) \varphi_p(z^{-1})}
\]  

(3.16)

where

\[
\bar{\sigma}^2 = \sigma^2 \prod_{j=1}^{q} H_j^2, \quad \theta'_q(z) = \prod_{j=1}^{q} (1 - H_j^{-1} z).
\]  

(3.17)

\(\gamma_x(z)\) in (3.16) can be viewed as the generating function of a process of the form

\[
\varphi_p(B)W_t = \theta'_q(B)\bar{u}_t = \left[ \prod_{j=1}^{q} (1 - H_j^{-1} B) \right] \bar{u}_t
\]  

(3.18)

while \(\gamma_x(z)\) in (3.14) is the generating function of

\[
\varphi_p(B)W_t = \theta_q(B)u_t = \left[ \prod_{j=1}^{q} (1 - H_j B) \right] u_t.
\]  

(3.19)

The processes (3.18) and (3.19) have the same autocovariance function and thus cannot be distinguished by looking at their seconds moments.
**Example 3.1** Identification of an ARMA(1, 1) model

\[ (1 - 0.5B)W_t = (1 - 0.2B)(1 + 0.1B)u_t \]  
(3.20)

\[ (1 - 0.5B)W_t = (1 - 5B)(1 + 10B)\tilde{u}_t \]  
(3.21)

have the same autocorrelation function.

In general, the models

\[ \varphi_p(B)W_t = \left[ \prod_{j=1}^{q} (1 - H_j^{\pm 1}B) \right] \tilde{u}_t \]  
(3.22)

all have the same autocovariance function (and are thus indistinguishable). Since it is easier with an invertible model, we select

\[ H^*_j = \begin{cases} H_j & \text{if } H_j < 1 \\ H^{-1}_j & \text{if } H_j > 1 \end{cases} \]  
(3.23)

where \(|H_j| \leq 1\), in order to have an invertible model.
3.3. Redundant parameters

Suppose $\phi_p(B)$ and $\theta_q(B)$ have a common factor, say $G(B)$:

$$\phi_p(B) = G(B)\phi_p(B), \quad \theta_q(B) = G(B)\theta_q(B). \quad (3.24)$$

Consider the models

$$\begin{align*}
\phi_p(B)W_t &= \theta_q(B)u_t \quad (3.25) \\
\phi_{p1}(B)W_t &= \theta_{q1}(B)u_t. \quad (3.26)
\end{align*}$$

The MA($\infty$) representations of these two models are

$$W_t = \psi(B)u_t, \quad (3.27)$$

where

$$\psi(B) = \frac{\theta_q(B)}{\phi_p(B)} = \frac{\theta_{q1}(B)\theta_q(B)G(B)}{\phi_{p1}(B)G(B)} = \frac{\theta_{q1}(B)G(B)}{\phi_{p1}(B)} = \psi_1(B), \quad (3.28)$$

$$W_t = \psi_1(B)u_t. \quad (3.29)$$

(3.25) and (3.26) have the same MA($\infty$) representation, hence the same autocovariance generating functions:

$$\gamma_x(z) = \sigma^2\psi(z)\psi(z^{-1}) = \sigma^2\psi_1(z)\psi_1(z^{-1}). \quad (3.30)$$

It is not possible to distinguish a series generated by (3.25) form one produced with (3.26). Among these two models, we will select the simpler one, i.e. (3.26). Further, if we tried to estimate (3.25) rather than (3.26), we would meet singularity problems (in the covariance matrix of the estimators).
4. Proofs and references

A general overview of the technique of generating functions is available in Wilf (1994).
References