1. Consider the density function
\[ \ell(y_1, \ldots, y_n; \theta) = \frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} (y_i - \theta)^2 \right\}. \]

(a) By modifying this density on a set of zero Lebesgue measure, show it is possible to make the maximum likelihood estimator equal to \( \sum_{i=1}^{n} y_i^4 \).

(b) Is it possible to preclude this type of manipulation? If so, how?

2. In maximum likelihood problems, show that:

(a) the maximum likelihood estimator may not exist;
(b) multiple maximum likelihood estimators may exist.

3. Let \( \ell(y; \theta), \theta \in \Theta \), be a likelihood function such that

(a) \( \Theta \) is a convex set and
(b) \( \log [\ell(y; \theta)] \) is a strictly concave function of \( \theta \).

Show that the maximum likelihood estimator of \( \theta \) is unique (if it does exist).

4. What happens to the maximum likelihood estimator when the model is reparameterized? Justify your answer.

5. Let \( (\mathcal{Y}, \mathcal{P}) \) where \( \mathcal{P} = (P_\theta = \ell(y; \theta) \cdot \mu, \theta \in \Theta) \), a dominated parametric model, and \( S(y) \) a sufficient statistic for \( \theta \).
(a) If $\lambda = g(\theta)$ is a one-to-one function (bijection) of $\theta$ and if the maximum likelihood estimator $\hat{\theta}(y)$ of $\theta$ is unique, how are the maximum likelihood estimators of $\lambda$ and $\theta$ related? Justify your answer.

(b) What happens when the maximum likelihood estimator of $\theta$ is not unique?

(c) Is the estimator $\hat{\theta}(y)$ a function of $S(y)$? Justify your answer.

6. Consider the equilibrium model:

$$q_t = ap_t + b + u_t,$$
$$S_t = \alpha p_t + \beta x_t + \nu_t,$$
$$q_t = S_t,$$

where $q_t$ is the quantity demanded, $p_t$ is the price, $S_t$ is quantity supplied, $x_t$ is an exogenous variable, and the vectors $(u_t, \nu_t)'$ for $t = 1, \ldots, n$ are independent with the same distribution

$$N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix} \right].$$

(a) Find the reduced form of this model.
(b) How are the parameters of this reduced form related to the structural form? Is this model underidentified, just identified, or overidentified?
(c) Find the maximum likelihood estimators of the reduced-form coefficients.
(d) Find the maximum likelihood estimators of the structural-form coefficients.

7. Give regularity conditions under which a sequence of maximum likelihood estimators converges almost surely to the true parameter value.

8. Consider the following assumptions:

H1: the variables $Y_1, \ldots, Y_n$ are independent and follow the same distribution with density $f(y; \theta)$, $\theta \in \Theta \subseteq \mathbb{R}^p$;

H2: the interior of $\Theta$ is non-empty, and $\theta_0$ belongs to the interior of $\Theta$;

H3: the true unknown value $\theta_0$ is identifiable;

H4: the log-likelihood

$$L_n(y; \theta) = \sum_{i=1}^{n} \log [f(y_i; \theta)]$$

is continuous in $\theta$;

H5: $E_{\theta_0} [\log f(Y_i; \theta)]$ is finite;
H6: the log-likelihood is such that $\frac{1}{n}L_n(y; \theta)$ converges almost surely to $E_{\theta_0}[\log(Y_i; \theta)]$ uniformly in $\theta \in \Theta$.

H7: the log-likelihood is twice continuously differentiable in open neighborhood of $\theta_0$;

H8: $I_1(\theta_0) = E_{\theta_0}\left[-\frac{\partial^2 \log f(Y; \theta)}{\partial \theta^2}\right]$ is finite and invertible.

If $\hat{\theta}_n$ is consistent sequence of local maxima, show that the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is $N[0, I_1(\theta_0)^{-1}]$.

9. Let $Y_1, \ldots, Y_n$ be a sample of independent identically distributed random variables with distribution $N[\mu, \sigma^2]$ where $\mu \neq 0$ and $\sigma > 0$. Find the asymptotic distribution of the maximum likelihood estimator of $\gamma = 1/\mu$.

10. Let $Y_1, \ldots, Y_n$ be a random sample of independent random variables from the exponential distribution with density:

$$f(y; \theta) = e^{-y-\theta}1_{y \geq \theta}.$$ 

(a) Which regularity condition is not satisfied in this problem?

(b) How does $\sqrt{n}(\hat{\theta}_n - \theta_0)$ behave for large $n$?

(c) Provide the asymptotic distribution of $\hat{\theta}_n$.

References