Foreasting of stationary and ARIMA processes *

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1. Wiener-Kolmogorov formula

According to Wold’s decomposition theorem, a second-order stationary process (with mean zero) can be written in the following form:

\[ X_t = Y_t + D_t \]

where

\[ Y_t = \sum_{j=0}^{\infty} d_j u_{t-j}, \quad d_0 = 1, \quad \{u_t\} \sim BB(0, \sigma^2), \]

\[ D_t \] is deterministic,

\[ u_t = X_t - P_L(X_t | X_{t-1}, X_{t-2}, \ldots). \]

Consequently,

\[ P_L(u_t | X_{t-1}, X_{t-2}, \ldots) = 0 \]

or, more generally,

\[ P_L(u_t | X_{t-\ell}, X_{t-\ell-1}, \ldots) = 0, \quad \forall \ell \geq 1. \]

If \( X_t \) is a strictly indeterministic process,

\[ X_t = \sum_{j=0}^{\infty} d_j u_{t-j}, \]

we have:

\[ P_L(X_t | X_{t-1}, X_{t-2}, \ldots) = P_L(u_t | X_{t-1}, X_{t-2}, \ldots) \]

\[ + P_L \left[ \sum_{j=1}^{\infty} d_j u_{t-j} | X_{t-1}, X_{t-2}, \ldots \right] \]

\[ = \sum_{j=1}^{\infty} d_j u_{t-j}. \]

If we furthermore suppose that the \( \{u_t\} \) are independent,

\[ E(u_t | X_{t-1}, X_{t-2}, \ldots) = E(u_t | u_{t-1}, u_{t-2}, \ldots) = 0, \]

\[ E(X_t | X_{t-1}, X_{t-2}, \ldots) = \sum_{j=1}^{\infty} d_j u_{t-j} \]

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We have the best prediction in the mean-square-error (MSE) sense. Let \( \{X_t\} \) a weakly stationary indeterministic process:

\[
X_t = \sum_{j=0}^{\infty} d_j u_{t-j} = d(B) u_t, \tag{1.1}
\]

where

\[
d_0 = 1, \quad d(B) = \sum_{j=0}^{\infty} d_j B^j. \tag{1.2}
\]

Let us denote:

\[
P_{t-j}X_t = P(X_t \mid X_{t-j}, X_{t-j-1}, \ldots).
\]

Then,

\[
P_{t-j}X_t = X_t \quad \text{for} \quad j \leq 0,
\]

\[
u_t = X_t - P(X_t \mid X_{t-1}, X_{t-2}, \ldots) = X_t - P_{t-1}X_t
\]

\[
P_{t-1}X_t = \sum_{j=0}^{\infty} d_j P_{t-1} u_{t-j} = \sum_{j=1}^{\infty} d_j u_{t-j}
\]

\[
= \left( \frac{d(B)}{B} \right)_+ u_{t-1}
\]

where we define

\[
\left( \sum_{j=-\infty}^{+\infty} h_j B^j \right)_+ = \sum_{j=0}^{\infty} h_j B^j,
\]

\[
\left[ \frac{d(B)}{B} \right]_+ = (d_0 B^{-1} + d_1 B^0 + d_2 B^1 + \cdots)
\]

\[
= \left( \sum_{j=0}^{\infty} d_j B^{j-1} \right)_+
\]

\[
= \sum_{j=1}^{\infty} d_j B^{j-1}. \tag{1.3}
\]
Similarly, we get at lag \( \ell \),

\[
P_{t-\ell}X_t = \sum_{j=0}^{\infty} d_j P_{t-\ell}u_{t-j} = \sum_{j=0}^{\infty} d_j u_{t-j}
\]

\[
= \left( \frac{d(B)}{B^\ell} \right)_+ u_{t-\ell}
\]

or, equivalently,

\[
P_tX_{t+\ell} = \left( \frac{d(B)}{B^\ell} \right)_+ u_t = \sum_{j=\ell}^{\infty} d_j u_{t+\ell-j}.
\]

(1.4)

The formula (1.4) is called the Wiener-Kolmogorov formula for linear prediction \( \ell \) periods ahead. We see that \( P_tX_{t+\ell} \) can be computed by dropping the most recent \( l \) terms from the Wold decomposition:

\[
X_t = \sum_{j=0}^{\infty} d_j u_{t-j}
\]

\[
= \sum_{j=0}^{l-1} d_j u_{t+\ell-j} + \sum_{j=\ell}^{\infty} d_j u_{t+\ell-j}
\]

\[
= e_t(l) + P_tX_{t+\ell}.
\]

(1.5)

If \( X_t \) is invertible, we can write

\[
u_t = \frac{1}{d(B)} X_t
\]

hence

\[
P_{t-\ell}X_t = \left( \frac{d(B)}{B^\ell} \right)_+ \frac{1}{d(B)} X_{t-\ell},
\]

\[
P_tX_{t+\ell} = \left( \frac{d(B)}{B^\ell} \right)_+ \frac{1}{d(B)} X_t, \ \ell \geq 1.
\]

1.1 Example For an AR(1) process,

\[
X_t = \varphi_1 X_{t-1} + u_t, \quad |\varphi_1| < 1
\]

(1.6)

we have the representation

\[
X_t = \frac{1}{1 - \varphi_1 B} u_t,
\]

(1.7)
hence

\[ P_{t-1}X_t = \varphi_1 P_{t-1}X_{t-1} + P_{t-1}u_t = \varphi_1 X_{t-1}, \tag{1.8} \]

\[ P_{t-\ell}X_t = \varphi_1 P_{t-\ell}X_{t-1} = \varphi_1 P_{t-\ell}(\varphi_1 X_{t-2} + u_{t-1}) = \varphi_1^2 P_{t-\ell}X_{t-2} = \varphi_1^\ell X_{t-\ell}. \]

If we use the Wiener-Kolmogorov formula, we get:

\[ P_{t-\ell}X_t = \left[ B^{-\ell} \frac{1}{1 - \varphi_1 B} \right] u_{t-\ell} = \left[ B^{-\ell} (1 + \varphi_1 B + \varphi_1^2 B^2 + \cdots) \right]_+ (1 - \varphi_1 B) X_{t-\ell} = \varphi_1^\ell (1 + \varphi_1 B + \varphi_1^2 B^2 + \cdots) (1 - \varphi_1 B) X_{t-\ell} = \frac{\varphi_1^\ell}{1 - \varphi_1 B} (1 - \varphi_1 B) X_{t-\ell} = \varphi_1^\ell X_{t-\ell}. \]

1.2 Example For an MA(1) process,

\[ X_t = (1 - \theta_1 B) u_t, \quad |\theta_1| < 1, \tag{1.9} \]

we have

\[ u_t = \frac{1}{1 - \theta_1 B} X_t, \tag{1.10} \]

hence the following forecasts: for \( \ell = 1 \),

\[ P_{t-1}X_t = \left[ B^{-1} (1 - \theta_1 B) \right]_+ \frac{1}{1 - \theta_1 B} X_{t-1} = \frac{-\theta_1}{1 - \theta_1 B} X_{t-1} = -\theta_1 \sum_{i=0}^{\infty} \theta_1^i X_{t-1-i} = -\sum_{i=1}^{\infty} \theta_1^i X_{t-i}; \tag{1.11} \]
Pour $\ell \geq 2$

$$P_{t-\ell}X_t = \left[ B^{-\ell} (1 - \theta_1 B) \right] + \frac{1}{1 - \theta_1 B} X_{t-\ell} = 0.$$ 

1.3 Example For an ARMA(1, 1) process of the form

$$(1 - \varphi_1 B) X_t = (1 - \theta_1 B) u_t,$$

$| \varphi_1 | < 1, \quad | \theta_1 | < 1,$

(1.12)

the Wold representation can be written:

$$X_t = \frac{(1 - \theta_1 B) u_t}{(1 - \varphi_1 B)} = d(B) u_t.$$ 

(1.13)

From the latter, we then get the following forecasts: at the horizon 1,

$$P_{t-1}X_t = \left[ \frac{d(B)}{B} \right] u_{t-1} = \left[ \frac{d(B)}{B} \right] + \frac{1}{d(B)} X_{t-1}$$ 

(1.14)

where

$$\left[ \frac{d(B)}{B} \right] = \left[ \frac{B^{-1} (1 - \theta_1 B)}{1 - \varphi_1 B} \right] +$$

$$= \left[ \frac{B^{-1}}{1 - \varphi_1 B} \frac{\theta_1}{1 - \varphi_1 B} \right] +$$

$$= B^{-1} \left[ 1 + \varphi_1 B \left( 1 + \varphi_1 B + \varphi_1^2 B^2 + \cdots \right) - \frac{\theta_1}{1 - \varphi_1 B} \right] +$$

$$= \left[ B^{-1} + \frac{\varphi_1}{1 - \varphi_1 B} - \frac{\theta_1}{1 - \varphi_1 B} \right] = \frac{(\varphi_1 - \theta_1)}{1 - \varphi_1 B}.$$ 

(1.15)

so that

$$P_{t-1}X_t = \frac{(\varphi_1 - \theta_1)}{1 - \varphi_1 B} \left( \frac{1 - \varphi_1 B}{1 - \theta_1 B} \right) X_{t-1}$$

$$= \frac{(\varphi_1 - \theta_1)}{1 - \theta_1 B} X_{t-1};$$ 

(1.16)

at lag $\ell$,

$$P_{t-\ell}X_t = \left[ \frac{d(B)}{B^\ell} \right] u_{t-\ell} = \left[ \frac{d(B)}{B^\ell} \right] + \frac{1}{d(B)} X_{t-\ell}$$ 

(1.17)
where
\[
\left[ \frac{d(B)}{B^\ell} \right]_+ = \left[ B^{-\ell} \frac{(1 - \theta_1 B)}{(1 - \varphi_1 B)} \right]_+ \\
= \left[ B^{-\ell} \frac{\theta_1 B^{-(\ell-1)}}{(1 - \varphi_1 B)} \right] \\
= \left[ \varphi_1^{\ell-1} \frac{\theta_1 \varphi_1^{\ell-1}}{1 - \varphi_1 B} \right] \\
= \frac{\varphi_1^{\ell-1} (\varphi_1 - \theta_1)}{1 - \varphi_1 B}, \tag{1.18}
\]

hence
\[
P_{t-\ell} X_t = \frac{(\varphi_1 - \theta_1) \varphi_1^{\ell-1}}{1 - \varphi_1 B} \left( 1 - \frac{\varphi_1 B}{1 - \theta_1 B} \right) X_{t-\ell} \\
= \frac{\varphi_1^{\ell-1} (\varphi_1 - \theta_1)}{1 - \theta_1 B} X_{t-\ell} \tag{1.19}
\]
or
\[
P_t X_{t+\ell} = \frac{\varphi_1^{\ell-1} (\varphi_1 - \theta_1)}{1 - \varphi_1 B} X_t.
\]

2. Chain rule for prediction

Let
\[
P_t X_{t+1} = \sum_{j=0}^{\infty} h_j X_{t-j}.
\]

Then,
\[
P_{t+\ell} X_{t+\ell+1} = \sum_{j=0}^{\infty} h_j X_{t+\ell-j} \\
= h_0 X_{t+\ell} + h_1 X_{t+\ell-1} + \cdots + h_\ell X_t + h_{\ell+1} X_{t-1} + \cdots
\]

\[
P_t [P_{t+\ell} X_{t+\ell+1}] = P_t X_{t+\ell+1} \\
= h_0 P_t X_{t+\ell} + h_1 P_t X_{t+\ell-1} + \cdots + h_{\ell-1} P_t X_{t+1} + h_\ell X_t + h_{\ell+1} X_{t-1} + \cdots
\]
\[
\ell - 1 \sum_{i=0}^{\ell-1} h_i P_t X_t+\ell-i + \sum_{i=\ell}^{\infty} h_i X_t+\ell-i = \\
\ell - 1 \sum_{i=0}^{\ell-1} h_i P_t X_t+\ell-i + \sum_{i=0}^{\infty} h_i+\ell X_t-i .
\]

This formula allows one to compute predictions several steps ahead from one-step ahead predictions.

3. Properties of prediction errors

Let

\[
P_t X_{t+\ell} = P \left( X_{t+\ell} \mid X_t, X_{t-1}, \ldots \right).
\]

If

\[
X_t = \sum_{j=0}^{\infty} d_j u_{t-j} ,
\]

where

\[
d_0 = 1, \quad u_t = X_t - P_{t-1}X_t ,
\]

\[
\text{V} (u_t) = \sigma^2 ,
\]

then

\[
X_{t+\ell} = \sum_{j=0}^{\infty} d_j u_{t+\ell-j} ,
\]

\[
P_t X_{t+\ell} = \sum_{j=\ell}^{\infty} d_j u_{t+\ell-j} .
\]

\[
e_t (\ell) \equiv X_{t+\ell} - P_t X_{t+\ell} = \\
\ell - 1 \sum_{j=0}^{\ell-1} d_j u_{t+\ell-j} = \\
u_{t+\ell} + d_1 u_{t+\ell-1} + \cdots + d_{\ell-1} u_{t+1}
\]
follows a $MA(\ell-1)$ process, hence
\[
V[e_t(\ell)] = \sigma^2 \left[ 1 + d_1^2 + \cdots + d_{\ell-1}^2 \right]
\]
\[
= \sigma^2 \sum_{i=0}^{\ell-1} d_i^2,
\]
where $d_0 = 1$. Consequently,
\[
e_t(1) = u_{t+1}.
\]
One-step ahead prediction errors associated with optimal are uncorrelated between each others. Further,
\[
E[e_t(\ell)e_{t+k}(\ell)] = \sigma^2 \sum_{i=0}^{\ell-k-1} d_id_{i+k}
\]
\[
= \sigma^2 \sum_{i=k}^{\ell-1} d_id_{i-k}, \quad 0 \leq k \leq \ell - 1,
\]
\[
E[e_t(\ell)e_{t+k}(\ell)] = \begin{cases} 
\sigma^2 \sum_{i=k}^{\ell-1} d_id_{i-k}, & \text{if } 0 \leq k \leq \ell - 1, \\
0, & \text{if } k \geq \ell,
\end{cases}
\]
\[
\text{Corr}[e_t(\ell), e_{t+k}(\ell)] = \begin{cases} 
\frac{\sigma^2 \sum_{i=k}^{\ell-1} d_id_{i-k}}{\sum_{j=0}^{\ell-1} d_j^2}, & \text{if } 0 \leq k \leq \ell - 1, \\
0, & \text{if } k \geq \ell,
\end{cases}
\]
\[
\text{Cov}[e_t(\ell), e_t(\ell+j)] = E[e_t(\ell)e_t(\ell+j)]
\]
\[
= \sigma^2 \sum_{i=0}^{\ell-1} d_id_{i+j} \neq 0, \quad \text{for } j \geq 0.
\]
The covariances between prediction errors at different horizons are given by:
\[
C[e_t(\ell), e_t(\ell+j)] = E[e_t(\ell)e_t(\ell+j)]
\]
\[
= \sigma^2 \sum_{i=0}^{\ell-1} d_id_{i+j}.
\]
4. Prediction with ARIMA models

Suppose $X_t$ is an ARIMA process of the form:

\[ \varphi_p(B) (1 - B)^d X_t = \theta_q(B) u_t + \bar{\mu}, \quad (4.1) \]

\[ u_t \sim BB(0, \sigma^2), \quad (4.2) \]

If we denote

\[ \varphi(B) = \varphi_p(B) (1 - B)^d, \quad (4.3) \]

we can write

\[ \varphi(B) X_t = \theta_q(B) u_t + \bar{\mu}, \quad (4.4) \]

or, equivalently,

\[ [1 - \varphi_1 B - \cdots - \varphi_{p+d} B^{p+d}] X_t = [1 - \theta_1 B - \cdots - \theta_q B^q] u_t + \bar{\mu}, \quad (4.5) \]

hence the difference-equation representation of $X_t$:

\[ X_t = \varphi_1 X_{t-1} + \cdots + \varphi_{p+d} X_{t-p-d} \\
+ u_t - \theta_1 u_{t-1} - \cdots - \theta_q u_{t-q} + \bar{\mu} \quad (4.6) \]

or

\[ X_t = \sum_{i=1}^{p+d} \varphi_i X_{t-i} - \sum_{j=1}^{q} \theta_j u_{t-j} + \bar{\mu} + u_t. \quad (4.7) \]

4.1 Example For an ARIMA$(1, 1, 0)$ process, we have:

\[ (1 - \varphi_1 B) (1 - B) X_t = u_t, \]

\[ [1 - \varphi_1 B + \varphi_2 B^2] X_t = [1 - (\varphi_1 + 1) B + \phi_1 B^2] X_t = u_t, \]

\[ \varphi_1 = (\varphi_1 + 1), \ \varphi_2 = -\phi_1. \]

On applying the projection operator $P_t$ on both sides of the equation (4.7), we obtain:

\[ P_t X_{t+1} = \sum_{i=1}^{p+d} \varphi_i P_t X_{t+1-i} - \sum_{j=0}^{q} \theta_j P_t u_{t+1-j} + \bar{\mu} \]

\[ = \sum_{i=1}^{p+d} \varphi_i X_{t+1-i} - \sum_{j=1}^{q} \theta_j u_{t+1-j} + \bar{\mu}, \]
\[ P_t X_{t+2} = \varphi_1 P_t X_{t+1} + \sum_{i=2}^{p+d} \varphi_i X_{t+2-i} - \sum_{j=2}^q \theta_j u_{t+2-j} + \bar{\mu}, \]

and, more generally,

\[ P_t X_{t+\ell} = \sum_{i=1}^{\ell-1} \varphi_i P_t X_{t+\ell-i} + \sum_{i=\ell}^{p+d} \varphi_i X_{t+\ell-i} - \sum_{j=\ell}^q \theta_j u_{t+\ell-j} + \bar{\mu}. \]

On noting that

\[ P_t u_{t+\ell} = \begin{cases} 0, & \text{if } \ell \geq 1, \\ u_{t+\ell}, & \text{if } \ell \leq 0, \end{cases} \]

we then see that

\[ P_t X_{t+\ell} = \sum_{i=1}^{p+d} \varphi_i P_t X_{t+\ell-i} + P_t u_{t+\ell} - \sum_{j=1}^q \theta_j P_t u_{t+\ell-j} + \bar{\mu} \quad (4.1) \]

and

\[ \varphi(B) P_t X_{t+\ell} = \theta(B) P_t u_{t+\ell} + \bar{\mu}, \]
\[ \varphi(B) X_{t+\ell} = \theta(B) u_{t+\ell} + \bar{\mu}, \]
\[ \varphi(B)(X_{t+\ell} - P_t X_{t+\ell}) = \theta(B)(u_{t+\ell} - P_t u_{t+\ell}). \]

Let

\[ \psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots = \sum_{j=0}^{\infty} \psi_j B^j, \]
\[ \varphi(B) \psi(B) = \theta(B), \]
\[ e_t(\ell) = X_{t+\ell} - P_t X_{t+\ell}, \quad \ell \geq 1, \]
\[ \varphi(B) e_t(\ell) = \varphi(B) \psi(B)(u_{t+\ell} - P_t u_{t+\ell}). \]

If we note that

\[ e_t(\ell) = 0, \quad \ell \leq 0 \]
\[ u_{t+\ell} - P_t u_{t+\ell} = \begin{cases} 0, & \ell \leq 0 \\ u_{t+\ell}, & \ell \geq 1 \end{cases} \]
we can simplify \( \varphi (B) \) on both sides and get

\[
e_t(\ell) = \psi (B)(u_{t+\ell} - P_t u_{t+\ell})
= u_{t+\ell} + \psi_1 u_{t+\ell-1} + \cdots + \psi_{\ell-1} u_{t+1}
= \sum_{i=0}^{\ell-1} \psi_i u_{t+\ell-i},
\]

where \( \psi_0 = 1 \). It follows that

\[
\begin{align*}
\mathbb{E}[e_t(\ell)] &= 0 \\
\mathbb{V}[e_t(\ell)] &= \mathbb{V}(\ell) = \sigma^2 \left[ 1 + \sum_{j=1}^{\ell-1} \psi_j^2 \right].
\end{align*}
\]

One-step ahead prediction errors \( e_t(1) \) are uncorrelated between each other. More generally,

\[
\begin{align*}
\mathbb{E}[e_t(\ell) e_{t-j}(\ell)] &= \left\{ \begin{array}{ll}
\sigma^2 \sum_{i=j}^{\ell-1} \psi_i \psi_{i-j}, & \text{if } 0 \leq j \leq \ell - 1, \\
0, & \text{if } | j | \geq \ell,
\end{array} \right.
\end{align*}
\]

(4.8)

\[
\begin{align*}
\mathbb{E}[e_t(\ell) e_{t}(\ell+j)] &= \left( \sum_{i=0}^{\ell-1} \psi_i \psi_{i+j} \right) \sigma^2.
\end{align*}
\]

If we assume that \( u_t \overset{iid}{\sim} N[0, \sigma_a^2] \)

\[
e_t(\ell) \sim N \left[ 0, \left\{ 1 + \sum_{j=1}^{\ell-1} \psi_j^2 \right\} \sigma^2 \right],
\]

we can compute confidence intervals for predictions:

\[
\mathbb{P} \left[ P_t X_{t+\ell} - c_{\alpha/2} \Delta_t \leq X_{t+\ell} \leq P_t X_{t+\ell} + c_{\alpha/2} \Delta_t \right] = 1 - \alpha
\]

where

\[
\Delta_t^2 = \left\{ 1 + \sum_{j=1}^{\ell-1} \psi_j^2 \right\} \sigma^2,
\]

(4.9)

\[
\mathbb{P} \left[ N(0, 1) \geq c_\alpha \right] = \alpha.
\]

If we compute predictions at different horizons \( \ell \), we obtain a prediction function:

\[
P_t X_{t+\ell} \equiv \hat{X}_t(\ell), \quad \ell = 1, 2, 3, \ldots,
\]
\[
\varphi(B)X_t = \theta(B)u_t + \bar{\mu},
\]
\[
\varphi(B)\hat{X}_t(\ell) = \theta(B)P_t u_{t+\ell} + \bar{\mu},
\]
\[
\varphi_p(B)(1-B)^d \hat{X}_t(\ell) = \theta(B)P_t u_{t+\ell} + \bar{\mu}.
\]

If \(d = 0\) and \(\ell\) is large, we have:
\[
\varphi_p(B)\hat{X}_t(\ell) \simeq \bar{\mu},
\]
\[
\hat{X}_t(\ell) \simeq \mu \equiv \frac{\bar{\mu}}{\varphi_p(B)} = \frac{\bar{\mu}}{1 - \varphi_1 - \cdots - \varphi_p}.
\]

If \(d = 1\) and \(\ell\) is large,
\[
\varphi_p(B)(1-B)\hat{X}_t(\ell) \simeq \bar{\mu}
\]
\[
(1-B)\hat{X}_t(\ell) \simeq \mu \equiv \frac{\bar{\mu}}{\varphi_p(B)}
\]
\[
\hat{X}_t(\ell) \simeq \mu_0 + \mu \ell \quad \text{Arithmetic progression.}
\]

If \(d = 2\) and \(\ell\) is large,
\[
\hat{X}_t(\ell) \simeq \mu_0 + \mu_1 \ell + \mu_2 \ell^2.
\]

5. Bibliographic notes

The reader will find general discussions of prediction based on ARIMA models in Box and Jenkins (1976, Sections 5.1-5.5, 5.7), Brockwell and Davis (1991, Sections 5.1-5.5) and Hamilton (1994, Chap. 4). On prediction for stationary processes and the Wiener-Kolmogorov formula, see also Wiener (1949), Whittle (1983), Whiteman (1983) and Sargent (1987).
References


