Solution to Econ 763 Assignment 2 (Winter 2017)
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Problem 1 (20 points)

Grading remarks: 5 points each for (a)–(d)

Suppose we have the formal series

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$$

where $\{u_t : t \in \mathbb{Z}\} \sim WN(0, \sigma^2)$. For a fixed t, we can in general write

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t-j} = \sum_{j=-\infty}^{\infty} Y_j = \sum_{j=-\infty}^{0} Y_j + \sum_{j=1}^{\infty} Y_j$$

where $Y_j \equiv \psi_j u_{t-j}$. In particular, the dependence of Y_j on t has been suppressed in the notation. Note that $E(Y_j) = 0$, $E(Y_j^2) = \psi_j^2 \sigma^2 < \infty$ so that $Y_j \in L_2$, and $E(Y_i Y_j) = 0$ for $i \neq j$.

(a) Convergence in mean of order 2

Proposition 4.2.6 in Dufour (2008b) implies that if

$$\infty > \sum_{j=-\infty}^{\infty} (E[Y_j^2])^{1/2} = \sum_{j=-\infty}^{\infty} (\psi_j^2 E(u_{t-j}^2))^{1/2} = \sigma \sum_{j=-\infty}^{\infty} |\psi_j| \tag{1}$$

then there exists random variables Y^- and Y^+ such that

$$\sum_{j=-m}^{0} Y_j \underset{m\to\infty}{\overset{2}{\to}} Y^-, \quad \sum_{j=0}^{n} Y_j \underset{n\to\infty}{\overset{2}{\to}} Y^+$$

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We can thus write $Y^- = \sum_{j=-\infty}^{0} Y_j$ and $Y^+ = \sum_{j=1}^{\infty} Y_j$. Moreover, we have

$$\sum_{j=-m}^{0} Y_j + \sum_{j=0}^{n} Y_j \underset{m,n\to\infty}{\overset{2}{\to}} Y^- + Y^+ \equiv Y.$$

Having shown convergence, we are now justified in writing

$$Y = \sum_{j=-\infty}^{\infty} Y_j = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}.$$

Remark: what the above has shown is that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ is sufficient for the convergence in mean of order 2 of $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$. A different result from Dufour (2008b) (Proposition 4.3.1) gives another sufficient condition

$$\infty > \sum_{j=-\infty}^{\infty} E[Y_t^2] = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j^2 \iff \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty. \tag{2}$$

We note that (1) is a strictly stronger condition than (2): the former implies the latter but the reverse implication fails. To see that, the convergence of $\sum_{j=-\infty}^{\infty} |\psi_j|$ implies that there is N sufficiently large that for $|n| \geq N$, we have $|\psi_j| < 1$. Then, for $n, m > N$, we have

$$\begin{aligned}\sum_{j=-m}^{n} \psi_j^2 &= \sum_{j=-N}^{N} \psi_j^2 + \sum_{j=N+1}^{n} \psi_j^2 + \sum_{j=-m}^{-N-1} \psi_j^2 \\ &\leq \sum_{j=-N}^{N} \psi_j^2 + \sum_{j=N+1}^{n} |\psi_j| + \sum_{j=-m}^{-N-1} |\psi_j|.\end{aligned}$$

When we let $m, n \to \infty$, absolute summability (i.e. (1)) implies that the second line above converges, which in turn gives the convergence of $\sum_{j=-\infty}^{\infty} \psi_j^2$. To see that square-summability doesn't imply absolute summability, consider

$$\psi_j = 0 \quad \forall j \leq 0, \quad \psi_j = \frac{1}{j} \quad \forall j \geq 1.$$

Then

$$\sum_{j=-\infty}^{\infty} \psi_j^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \sum_{j=-\infty}^{\infty} |\psi_j| = \sum_{j=1}^{\infty} \frac{1}{j} = +\infty.$$

(b) Convergence in mean of order r

Following the same approach above and Proposition 4.2.6 (Dufour (2008b)), we may infer that for $r \geq 1$, the condition

$$\infty > \sum_{j=-\infty}^{\infty} (E[|\psi_j u_{t-j}|^r])^{1/r} = E(|u_t|^r)^{1/r} \sum_{j=-\infty}^{\infty} |\psi_j| \iff \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \tag{3}$$

is sufficient for $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ to converge in mean of order r. Of course, here *we also need each u_t to be in L_r.*

For $r < 1$, we also appeal to Proposition 4.2.6. To be specific, that proposition tells us that for $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ to converge in mean (i.e. in L_1), it also suffices to have $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ (and that for each t, $E(|u_t|)$ is finite but this follows because $E(u_t^2)$ is finite.) But convergence in L_1 implies convergence in L_r for $r < 1$, so the same sufficient condition is enough for $\sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$ to converge in mean of order $r < 1$.

(c) Almost sure convergence

Proposition 4.2.6 again gives us a sufficient condition

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \tag{4}$$

Proposition 4.3.1 competes to give another sufficient condition

$$\sum_{j=1}^{\infty} (\log j)^2 \psi_j^2 < \infty, \quad \sum_{j=-\infty}^{-1} (\log(-j))^2 \psi_j^2 < \infty \tag{5}$$

We see that (5) does not imply (4). For example, when $\psi_j = 0$ for $j \leq 0$ and $\psi_j = \frac{1}{j}$ for $j \geq 1$, we have

$$\sum_{j=1}^{\infty} (\log j)^2 \psi_j^2 = \sum_{j=1}^{\infty} \frac{(\log j)^2}{j^2} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |\psi_j| = \sum_{j=1}^{\infty} \frac{1}{j} \to \infty.$$

(d) Convergence in probability

Since convergence in probability is implied by convergence in mean of order r ($r > 0$) and almost sure convergence, each of the conditions (2), (4), and (5) will be sufficient here.

Problem 2 (10 points)

Grading remarks: 5 points each for (a) and (b)

Consider an $MA(1)$ model

$$X_t = \bar{\mu} + u_t - \theta u_{t-1}, \quad t \in \mathbb{Z}$$

where $u_t \sim WN(0, \sigma^2)$ and $\sigma^2 > 0$.

(a) *The first autocorrelation of this model cannot be greater than* 0.5 *in absolute value.*

Proof. We have

$$\operatorname{Cov}(X_t, X_{t+1}) = E[(u_t - \theta u_{t-1})(u_{t+1} - \theta u_t)] = -\theta E(u_t^2) = -\theta\sigma^2,$$
$$\operatorname{Var}(X_t) = \operatorname{Var}(u_t) + \theta^2 \operatorname{Var}(u_{t-1}) = (1+\theta^2)\sigma^2.$$

This implies

$$|\rho(1)| = \left|\frac{\operatorname{Cov}(X_t, X_{t+1})}{\operatorname{Var}(X_t)}\right| = \frac{|\theta|}{1+\theta^2}$$

which is less than or equal to $\frac{1}{2}$ because

$$\frac{|\theta|}{1+\theta^2} \leq \frac{1}{2} \iff 2|\theta| \leq 1+\theta^2 \iff (|\theta|-1)^2 \geq 0.$$

□

(b) *Values of the model parameters for which this upper bound is attained.*

Answer. As shown in (a), we have

$$2(1+\theta^2)\left(\frac{1}{2} - |\rho(1)|\right) = (1+\theta^2)(|\theta|-1)^2 \geq 0$$

which equals 0 iff $|\theta| = 1$. That is, when $\theta = \pm 1$, the absolute value of the first autocorrelation equals $\frac{1}{2}$. □

Problem 3 (72 points)

Grading remarks: for each process, 3 points each for (a)–(f) and $4 \times 18 = 72$ points total

Let $\{X_t : t \in \mathbb{Z}\}$ be an $MA(q)$ process. For $q = 3, 4, 5, 6$, check whether the following inequalities are correct:

(a) $|\rho(1)| \leq 0.75$;

(b) $|\rho(2)| \leq 0.90$;

(c) $|\rho(3)| \leq 0.90$;

(d) $|\rho(4)| \leq 0.90$;

(e) $|\rho(5)| \leq 0.90$;

(f) $|\rho(6)| \leq 0.90$.

A general $MA(q)$ process can be written as

$$X_t = \mu + u_t + \sum_{t=1}^{q} \theta_j u_{t-j} = \mu + \theta(L) u_t \quad \text{with} \quad \theta(L) = 1 + \theta_1 L - \ldots + \theta_q L^q.$$

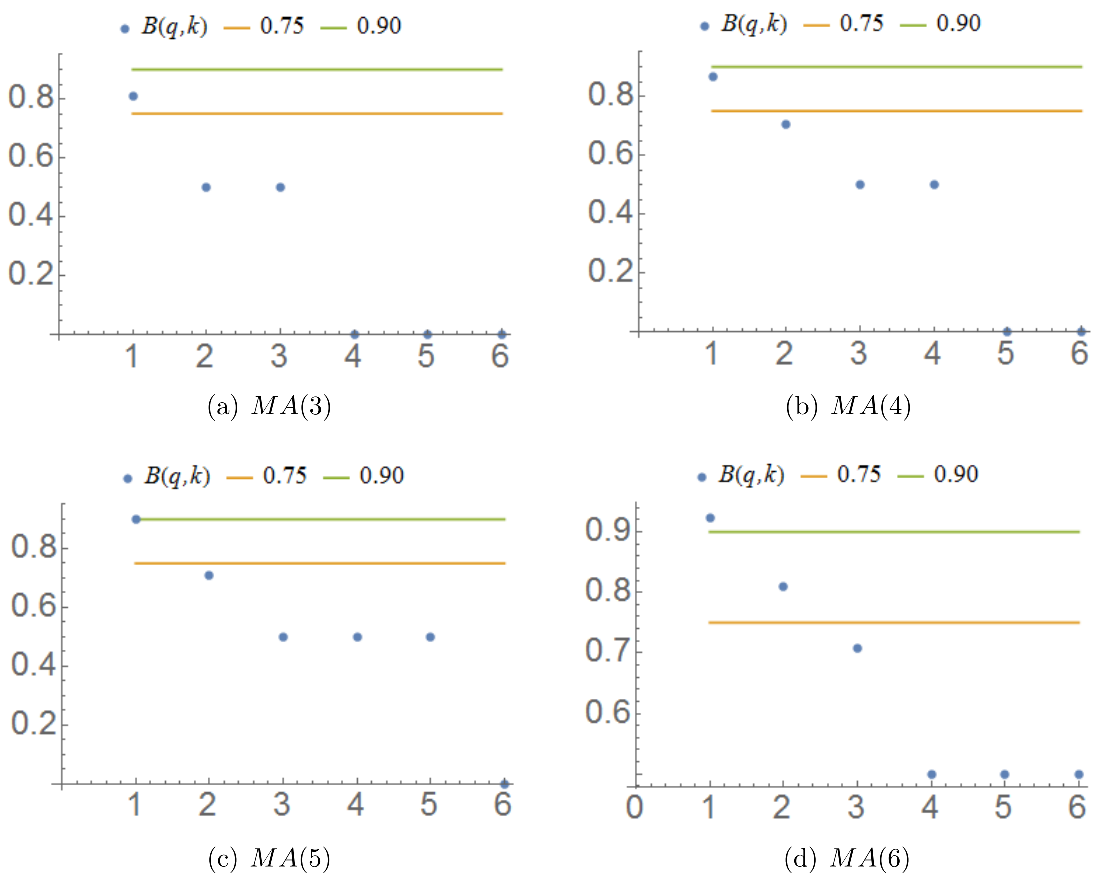
From the lecture notes (Dufour (2008a)), the autocorrelation coefficients can be computed as follows

$$\begin{aligned} \rho(k) &= \left(\theta_k + \textstyle\sum_{j=1}^{q-k} \theta_j \theta_{j+k}\right) / \left(1 + \textstyle\sum_{j=1}^{q} \theta_j^2\right), & 1 \le k \le q \\ &= 0, & k \ge q+1. \end{aligned}$$

In particular, the autocorrelations vanish for $k \ge q+1$. Moreover, formula (6.12) from the lecture notes gives us

$$|\rho(k)| \le B(q,k) \equiv \cos\left(\frac{\pi}{\lfloor q/k \rfloor + 2}\right).$$

Plotting $B(q,k)$ for various q and k gives



(a) $MA(3)$ (b) $MA(4)$

(c) $MA(5)$ (d) $MA(6)$

Figure 1: Upperbounds for autocorrelations of some MA processes

MA(3)

From the Figure 1, we know that (b)–(f) must hold, but let's verify this algebraically. Because $\rho(k) = 0$ for $k \geq 4$, the inequalities in (d)–(f) hold automatically. For (a)–(c), we write

$$\begin{aligned}\rho(1) &= \frac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2},\\ \rho(2) &= \frac{\theta_2 + \theta_1\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2},\\ \rho(3) &= \frac{\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}.\end{aligned}$$

From these, (c) holds because

$$2|\theta_3| \leq 1 + \theta_3^2 \leq 1 + \theta_1^2 + \theta_2^2 + \theta_2^3 \implies |\rho(3)| \leq \frac{1}{2} < 0.90.$$

In a similar manner, we can use the inequality $2ab \leq a^2 + b^2$ to infer

$$2|\theta_2 + \theta_1\theta_3| \leq 2|\theta_2| + |\theta_1||\theta_3| \leq 1 + \theta_2^2 + \theta_1^2 + \theta_3^2 \implies |\rho(2)| \leq \frac{1}{2} < 0.90.$$

So (b) indeed holds. As the figure suggest however, (a) can fail. And it does when we set $\theta_1 = \theta_2 = \theta_3 = \theta = \frac{3}{2}$ so that

$$|\rho(1)| = \frac{\theta(1 + 2\theta)}{1 + 3\theta^2} = \frac{24}{31} > \frac{24}{32} = 0.75.$$

MA(4)

Again, Figure 1 says that (b)–(f) are true whereas (a) may fail. For (e)–(f), the implications are immediate because $\rho(5) = \rho(6) = 0$. For the rest, we write

$$\begin{aligned}\rho(1) &= \frac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2},\\ \rho(2) &= \frac{\theta_2 + \theta_1\theta_3 + \theta_2\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2},\\ \rho(3) &= \frac{\theta_3 + \theta_1\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2},\\ \rho(4) &= \frac{\theta_4}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}.\end{aligned}$$

Because $|\theta_4| \leq \frac{1}{2}(1 + \theta_4^2)$, it's obvious that $|\rho(4)| \leq \frac{1}{2} < 0.90$. Similarly, $|\theta_3| \leq \frac{1}{2}(1 + \theta_3^2)$ and $|\theta_1||\theta_4| \leq \frac{1}{2}(\theta_1^2 + \theta_4^2)$ imply that $|\rho(3)| \leq \frac{1}{2} < 0.90$. To prove $|\rho(2)| \leq 0.90$ we can assume WLOG that $\theta_i \geq 0$ so that $|\rho(2)| \leq 0.90$ is equivalent to

$$9 + 9\theta_1^2 + 9\theta_2^2 + 9\theta_3^2 + 9\theta_4^2 \geq 10\theta_2 + 10\theta_1\theta_3 + 10\theta_2\theta_4.$$

Noting that $9\theta_1^2 + 9\theta_3^2 - 10\theta_1\theta_3 = (2\theta_1)^2 + (2\theta_2)^2 + 5(\theta_1 - \theta_3)^2$, we only need to prove

$$9 + 9\theta_2^2 + 9\theta_4^2 \geq 10\theta_2 + 10\theta_2\theta_4. \tag{$*$}$$

We can treat $(*)$ as an inequality for θ_4 equals some fixed $y \geq 0$ and while $\theta_2 = x \geq 0$ is allowed to vary. That is, $(*)$ follows if we can show that

$$9 + 9x^2 + 9y^2 \geq 10x + 10xy = (10 + 10y)x \tag{$**$}$$

for all $x, y \geq 0$. With $y \geq 0$ fixed, the LHS above is convex in x whereas the RHS is linear. The derivative (w.r.t. to x) of the RHS is $10 + 10y$ whereas the derivative of the LHS is $18x$ which equals $9 + 10y$ when $x = \frac{5}{9}(1 + y)$. At this value of x, $(**)$ is equivalent to

$$81 + 81y^2 + 25(y+1)^2 \geq 50(y+1)^2 \iff 0 \leq 56y^2 - 50y + 56$$

which is true because $56y^2 - 50y + 56 = 31y^2 + 31 + 25(y-1)^2$. Due to the convexity observation from the previous paragraph, that the inequality holds for $x = \frac{5}{9}(1 + y)$ is enough for $(**)$ to hold for all $x \geq 0$ given y is fixed. As y is arbitrary, $(**)$ and, thus, $(*)$ must hold generally. In other words, $|\rho(2)| \leq \frac{9}{10}$ is true.

Finally, (a) fails when we set $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta$ so that

$$|\rho(1)| = \frac{\theta + 3\theta^2}{1 + 4\theta^2} = \frac{232}{281} > \frac{210}{280} = 0.75.$$

$MA(5)$

Figure 1 says that (b)–(f) are true while (a) may fail. $\rho(6) = 0$ so (f) is immediate. For (a)–(e), we write

$$\rho(1) = \frac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4 + \theta_4\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},$$
$$\rho(2) = \frac{\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \theta_3\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},$$
$$\rho(3) = \frac{\theta_3 + \theta_1\theta_4 + \theta_2\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},$$
$$\rho(4) = \frac{\theta_4 + \theta_1\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2},$$
$$\rho(5) = \frac{\theta_5}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2}.$$

Using the same techniques as above, we can again show that $|\rho(3)|$, $|\rho(4)|$, and $|\rho(5)|$ are less than or equal to $\frac{1}{2} < 0.90$. The autocorrelation of order 1 $\rho(1)$ can exceed 0.75 in absolute value: when $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta = \frac{3}{2}$, we have

$$|\rho(1)| = \frac{\theta + 4\theta^2}{1 + 5\theta^2} = \frac{6}{7} > 0.75.$$

As with the $MA(4)$ case, $\rho(2)$ poses a more challenging problem. The algebra seems intimidating so we settle with formula (6.12) from Dufour (2008a):

$$|\rho(2)| \leq \cos\left(\frac{\pi}{\lfloor q/k \rfloor + 2}\right)\Bigg|_{q=5,k=2} = \frac{1}{\sqrt{2}} < 0.9.$$

$MA(6)$

We get no free lunch with this one as none of the correlation is 0. The standard formulas give

$$\begin{aligned}
\rho(1) &= \frac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4 + \theta_4\theta_5 + \theta_5\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(2) &= \frac{\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \theta_3\theta_5 + \theta_4\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(3) &= \frac{\theta_3 + \theta_1\theta_4 + \theta_2\theta_5 + \theta_3\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(4) &= \frac{\theta_4 + \theta_1\theta_5 + \theta_2\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(5) &= \frac{\theta_5 + \theta_1\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}, \\
\rho(6) &= \frac{\theta_6}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2}.
\end{aligned}$$

A quick glance gives $|\rho(4)|$, $|\rho(5)|$ and $|\rho(6)|$ are no more than $\frac{1}{2} < 0.9$. To find the counter example for $\rho(1)$, we set $\theta_1 = \ldots = \theta_6 = \theta = \frac{8}{5}$ to obtain

$$|\rho(1)| = \frac{\theta + 5\theta^2}{1 + 6\theta^2} = \frac{360}{409} > \frac{360}{480} = 0.75.$$

The algebra for $\rho(2)$ and $\rho(3)$ looks scary so we again use (6.12) from Dufour (2008a):

$$\begin{aligned}
|\rho(2)| &\leq \cos\left(\frac{\pi}{\lfloor q/k \rfloor + 2}\right)\Bigg|_{q=6,k=2} = \frac{1}{4}(1 + \sqrt{5}) < 0.9. \\
|\rho(3)| &\leq \cos\left(\frac{\pi}{\lfloor q/k \rfloor + 2}\right)\Bigg|_{q=6,k=3} = \frac{1}{\sqrt{2}} < 0.9.
\end{aligned}$$

Problem 4 (300 points)

Grading remarks: for each process, 2 points for (a), 2 points for (b), 7 $(1+4+2)$ points for (c), 3 points for (d), 5 points for (e), 2 points for (f), 5 $(2+3)$ points for (g), 4 points for (h), and so $6 \times 30 = 180$ points total

Some general results for $ARMA(p,q)$ (p, q finite)

For some finite and positive integers p and q, we consider a process $\{X_t : t \in \mathbb{Z}\}$ which satisfies the equation

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j} \tag{6}$$

where $\{u_t : t \in \mathbb{Z}\}$ is a homoskedastic white noise with common variance σ^2. Using operational notation, we can define $\varphi(B) = 1 - \sum_{j=1}^{p} \varphi_j B^j$ and $\theta(B) = 1 - \sum_{j=1}^{q} \theta_j B^j$ and write

$$\varphi(B)X_t = \bar{u} + \theta(B)u_t. \tag{7}$$

(1) **Stationarity condition:** if the polynomial $\varphi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p$ has all its roots outside the unit circle, the equation (6) has one and only one weakly stationary solution, which can be written

$$X_t = \mu + [\varphi(B)]^{-1}\theta(B)u_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j} \tag{8}$$

where

$$\mu = \frac{\bar{\mu}}{\varphi(B)} = \frac{\bar{\mu}}{1 - \sum_{j=1}^{p} \varphi_j},$$
$$\frac{\theta(B)}{\varphi(B)} \equiv \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j.$$

(2) The ψ_j coefficients are obtained by solving the equation $\varphi(B)\psi(b) = \theta(B)$:

$$\left(1 - \sum_{k=1}^{p} \varphi_k B^k\right)\left(\sum_{j=0}^{\infty} \psi_j B^j\right) = 1 - \sum_{j=1}^{q} \theta_j B^j \tag{9}$$

and comparing powers of B's on both sides. For examples, (below we define $\theta_0 = -1$)

$$\begin{aligned} \psi_0 &= -\theta_0 = 1, \\ \psi_1 - \varphi_1 &= -\theta_1, \\ \psi_2 - \varphi_1\psi_1 - \varphi_2 &= -\theta_2, \\ \vdots \; \vdots \; \vdots \;& \vdots \; \vdots \\ \psi_j - \sum_{k=1}^{j} \varphi_k \psi_{j-k} &= -\theta_j, \quad (j = 0, 1, \ldots, q) \end{aligned}$$

If we define $\psi_j = 0$ for $j < 0$ then the last line above can be rewritten as $\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} = -\theta_j$ for $j = 0, \ldots, q$. For $j > q$, things get slightly trickier. The advantage of this re-expression is that for $j > q$, we can also write $\psi_j - \sum_{k=1}^{p} \varphi_k \psi_{j-k} = 0$.

Thus, a convenient algorithm for solving for ψ_j is that:

(i) define $\psi_{-p} = \psi_{-(p-1)} = \ldots = \psi_{-1} = 0$,

(ii) for $j = 0, 1, \ldots, q$, recursively compute $\psi_j = -\theta_j + \sum_{k=1}^{p} \varphi_k \psi_{j-k}$,

(iii) for $j > q$, continue the recursion $\psi_j = \sum_{k=1}^{p} \varphi_k \psi_{j-k}$.

(c) **Invertibility:** If the ARMA process (7) is second-order stationary, then the process $\{X_t\}$ satisfies an equation of the form

$$\sum_{j=0}^{\infty} \tilde{\phi}_j X_{t-j} = \tilde{\mu} + u_t$$

iff the roots of the polynomial $\theta(B)$ are outside the unit circle. Further, when the representation above exists, we have

$$\tilde{\phi}(B) = \theta(B)^{-1}\varphi(B), \quad \tilde{\mu} = \theta(B)^{-1}\bar{\mu} = \frac{\bar{\mu}}{1 - \sum_{j=1}^{q} \theta_j}.$$

In particular, any stationary $AR(p)$ process is invertible. Note that invertibility is actually a separate concept from stationarity. In Box et al. (2008), a linear process $X_t = \mu + \sum_{j=1}^{\infty} \psi_j a_{t-j}$ is invertible if $\sum_{j=0}^{\infty} |\pi_j| < \infty$, where $\pi(B) = \psi^{-1}(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j$.

(d) **Autocovariances and autocorrelations:** Suppose that

(i) the polynomial $\varphi(z)$ has is roots outside the unit circle and the process X_t the unique stationary solution to $\varphi(B)X_t = \bar{u} + \theta(B)u_t$,

(ii) $E(X_{t-j}u_t) = 0$ for all $j \geq 1$.

By the stationarity assumption, $E(X_t) = \mu$ for some μ and for all t. This μ satisfies

$$\mu = E(X_t), \forall t \implies \varphi(B)\mu = E[\varphi(B)X_t] = \bar{u} \implies \mu = \frac{\bar{\mu}}{1 - \sum_{j=1}^{p} \varphi_j}.$$

Now, let us define $Y_t = X_t - \mu$ so that $E(Y_t) = 0$ and $\varphi(B)Y_t = \theta(B)u_t$. It follows that for $k > 0$

$$Y_{t+k} = \sum_{j=1}^{p} \varphi_j Y_{t+k-j} + u_{t+k} - \sum_{j=1}^{q} \theta_j u_{t+k-j},$$

$$\implies E[Y_t Y_{t+k}] = \sum_{j=1}^{p} \varphi_j E[Y_t Y_{t+k-j}] + E[Y_t u_{t+k}] - \sum_{j=1}^{q} \theta_j E[Y_t u_{t+k-j}],$$

which implies

$$\gamma(k) = \sum_{j=1}^{p} \varphi_j \gamma(k-j) - \sum_{j=1}^{q} \theta_j \gamma_{xu}(k-j) \tag{10}$$

where

$$\gamma_{xu}(k) = E(Y_t u_{t+k}) = \begin{cases} 0 & \text{if } k \geq 1 \\ \sigma^2 & \text{if } k = 0 \end{cases}$$

and $\gamma_{xu}(k) \neq 0$ in general for $k \leq 0$. That is, for $1 \leq k \leq q$,

$$\begin{aligned} \gamma_{xu}(-k) &= E(Y_t u_{t-k}) \\ &= E\left[\left(\sum_{j=1}^{p} Y_{t-j} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}\right) u_{t-k}\right] \\ &= \sum_{j=1}^{p} \gamma_{xu}(-k+j) - \theta_k \sigma^2. \end{aligned}$$

As j in the last line above is strictly positive, $-k+j > -k$ so that γ_{xu} can be computed backwards recursively. Once we have found γ_{xu}, we can solve (10) and

$$\gamma(0) = \sum_{j=1}^{p} \varphi_j \gamma(j) + \sigma^2 - \sum_{j=1}^{q} \theta_j \gamma_{xu}(-j)$$

for $\gamma(0), \gamma(1), \ldots, \gamma(p)$ in terms of the ARMA coefficients. Then for $k > p$, $\gamma(k)$ can be computed using (10). Finally, the autocorrelation $\rho(0)$ is simply $\frac{\gamma(k)}{\gamma(0)}$.

(e) **Partial autocorrelations**: the partial autocorrelation of order k, denoted by $\phi(k)$, is computed as follows: first, we define

$$\Phi(k) \equiv \begin{pmatrix} 1 & \rho(1) & \ldots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & \ldots & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho(k-2) & \rho(k-3) & \ldots & 1 & \rho(1) \\ \rho(k-1) & \rho(k-2) & \ldots & \rho(1) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k-1) \\ \rho(k) \end{pmatrix}. \tag{11}$$

Then, $\phi(k)$ is just the k-th entry of $\Phi(k)$.

The $AR(1)$ process $X_t = 0.5X_{t-1} + u_t$

Write this as $(1-\varphi_1)X_t = \bar{u} + u_t$ where $\varphi_1 = 0.5$ and $\bar{u} = 0$. Here, $u_t \sim N(0, \sigma^2)$ where $\sigma = 1$.

(a) The process is stationary because $\varphi(z) = 1 - 0.5z$ has root $z = 2$ which is outside the unit circle.

(b) The process is invertible, as is any other $AR(p)$ process for some finite p.

(c) (i) $E(X_t) = \frac{\bar{u}}{1-\varphi_1} = \frac{0}{1-0.5} = 0$,

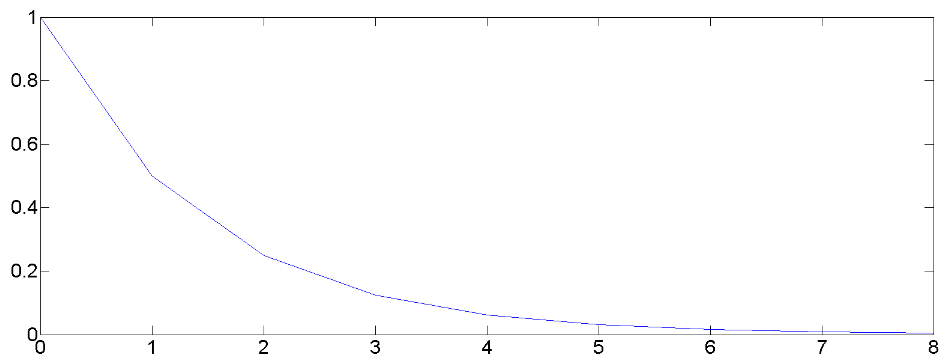
(ii) Using formulae (7.47) and (7.46) from Dufour (2008a), we have

$$\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1-\varphi_1^2} = \frac{1}{0.75} = \frac{4}{3};$$
$$\gamma(k) = \varphi_1^k \gamma(0) = \frac{4}{2^k 3}.$$

(iii) The autocorrelations are

$$\rho(0) = 1, \quad \rho(k) = \varphi_1^k = \frac{1}{2^k}.$$

(d) We can plot $\rho(k)$ for $k = 0, \ldots, 8$:



(e) Write $MA(\infty)$ representation as $X_t = \psi(B)u_t$ where $\psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j$. Because

$$\frac{1}{1-\varphi_1 B} = 1 + \varphi_1 B + \varphi_1^2 B^2 + \varphi_3^2 B^3 + \varphi_4^2 B^4 + \cdots,$$

we have

$$\psi_0 = 1,$$
$$\psi_1 = \varphi_1 = 0.5;$$
$$\psi_2 = \varphi_1^2 = 0.25;$$
$$\psi_3 = \varphi_1^3 = 0.125;$$
$$\psi_4 = \varphi_1^4 = 0.0625.$$

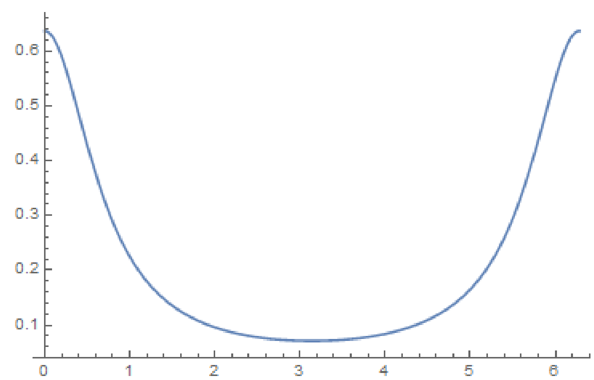
(f) With $\psi(z) = \frac{1}{1-\varphi_1 z}$ as defined above, then

$$\begin{aligned}\gamma_x(z) &= \sigma^2\psi(z)\psi(z^{-1}) \\ &= \frac{\sigma^2}{(1-\varphi_1 z)(1-\varphi_1 z^{-1})} \\ &= \frac{1}{(1-0.5z)(1-0.5/z)} \\ &= \frac{4z}{(2-z)(2z-1)}.\end{aligned}$$

(g) By Proposition 11.14 from Dufour (2008a), we have

$$\begin{aligned}f_x(\omega) &= \frac{\sigma^2}{2\pi}\psi(\exp(i\omega))\psi(\exp(-i\omega)) \\ &= \frac{\sigma^2}{2\pi}\frac{1}{(1-\varphi_1\exp(i\omega))(1-\varphi_1\exp(-i\omega))} \\ &= \frac{1}{2\pi[1-0.5\exp(i\omega)][1-0.5\exp(-i\omega)]} \\ &= \frac{2}{\pi(5-4\cos(\omega))}.\end{aligned}$$

Plotting it yields:



(h) Using the formula (11) four times, we get

$$\phi(1) = \frac{1}{2}, \quad \phi(2) = \phi(3) = \phi(4) = 0.$$

The $AR(1)$ process $X_t = 10 - 0.75X_{t-1} + u_t$

Write this as $(1-\varphi_1)X_t = \bar{u} + u_t$ where $\varphi_1 = -0.75$ and $\bar{u} = 10$. Here, $u_t \sim N(0, \sigma^2)$ where $\sigma = 1$.

(a) The process is stationary because $\varphi(z) = 1 + 0.75z$ has root $z = -\frac{4}{3}$ which is outside the unit circle.

(b) The process is invertible, as is any other $AR(p)$ process for some finite p.

(c) (i) $E(X_t) = \frac{\bar{u}}{1-\varphi_1} = \frac{10}{1+0.75} = \frac{40}{7}$,

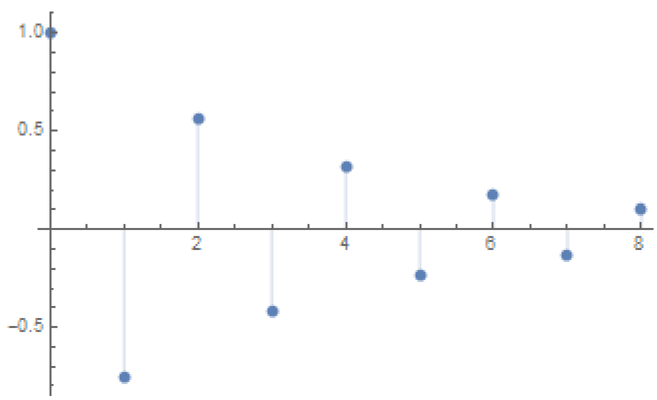
(ii) Using formulae (7.47) and (7.46) from Dufour (2008a), we have

$$\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1-\varphi_1^2} = \frac{16}{7};$$
$$\gamma(k) = \varphi_1^k \gamma(0) = \frac{(-3)^k 16}{4^k 7}.$$

(iii) The autocorrelations are

$$\rho(0) = 1, \quad \rho(k) = \varphi_1^k = \frac{(-3)^k}{4^k}.$$

(d) We can plot $\rho(k)$ for $k = 0, \ldots, 8$:



(e) Write $MA(\infty)$ representation as $X_t = \psi(B)u_t$ where $\psi(B) = \varphi(B)^{-1} = \sum_{j=0}^{\infty} \psi_j B^j$. Because

$$\frac{1}{1-\varphi_1 B} = 1 + \varphi_1 B + \varphi_1^2 B^2 + \varphi_3^2 B^3 + \varphi_4^2 B^4 + \cdots,$$

we have

$$\psi_0 = 1,$$
$$\psi_1 = \varphi_1 = \frac{-3}{4};$$
$$\psi_2 = \varphi_1^2 = \frac{9}{16};$$
$$\psi_3 = \varphi_1^3 = \frac{-27}{64};$$
$$\psi_4 = \varphi_1^4 = \frac{81}{256}.$$

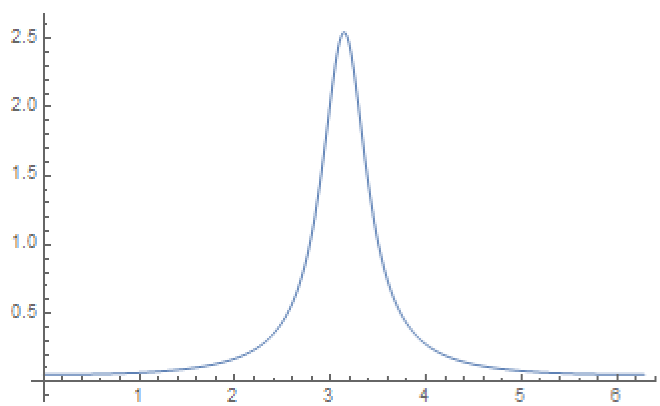
(f) With $\psi(z) = \frac{1}{1-\varphi_1 z}$ as defined above, then

$$\begin{aligned}\gamma_x(z) &= \sigma^2\psi(z)\psi(z^{-1}) \\ &= \frac{\sigma^2}{(1-\varphi_1 z)(1-\varphi_1 z^{-1})} \\ &= \frac{16z}{12+25z+12z^2}.\end{aligned}$$

(g) By Proposition 11.14 from Dufour (2008a), we have

$$\begin{aligned}f_x(\omega) &= \frac{\sigma^2}{2\pi}\psi(\exp(i\omega))\psi(\exp(-i\omega)) \\ &= \frac{\sigma^2}{2\pi}\frac{1}{(1-\varphi_1\exp(i\omega))(1-\varphi_1\exp(-i\omega))} \\ &= \frac{1}{2\pi[1+0.75\exp(i\omega)][1+0.75\exp(-i\omega)]} \\ &= \frac{1}{\pi(3.125+3\cos(\omega))}.\end{aligned}$$

Plotting it yields:



(h) Using the formula (11) four times, we get

$$\phi(1) = -\frac{3}{4}, \quad \phi(2) = \phi(3) = \phi(4) = 0.$$

The $AR(2)$ process $X_t = 10 + \frac{7}{10}X_{t-1} - \frac{1}{5}X_{t-2} + u_t$

Write this as

$$(1 - \varphi_1 B - \varphi_2 B^2)X_t = \bar{\mu} + u_t, \quad \varphi_1 = \frac{7}{10}, \quad \varphi_2 = \frac{-1}{5} \cdot .$$

As before, $u_t \sim N(0, \sigma^2)$ with $\sigma = 1$.

(a) Stationarity holds because $1 - \varphi_1 z - \varphi_2 z^2$ have 2 complex roots that both are outside the unit circle.

(b) Invertibility is immediate because this is an $AR(2)$ process.

(c) Using the formulas (7.49–51) from Dufour (2008a), we have:

(i) $\frac{\bar{\mu}}{1-\varphi_1-\varphi_2} = 20$;

(iii)

$$\begin{aligned}
\rho(0) &= 1; \\
\rho(1) &= \frac{\varphi_1}{1-\varphi_2} = \frac{7}{12}, \\
\rho(2) &= \frac{\varphi_1^2 + \varphi_2(1-\varphi_2)}{1-\varphi_2} = \frac{5}{24}, \\
\rho(3) &= \varphi_1\rho(2) + \varphi_2\rho(1) = \frac{7}{240}, \\
\rho(4) &= \varphi_1\rho(3) + \varphi_2\rho(2) = \frac{-17}{800}, \\
\rho(5) &= \varphi_1\rho(4) + \varphi_2\rho(3) = \frac{-497}{24000}, \\
\rho(6) &= \varphi_1\rho(5) + \varphi_2\rho(4) = \frac{-2459}{240000}, \\
\rho(7) &= \varphi_1\rho(6) + \varphi_2\rho(5) = \frac{-7273}{2400000}, \\
\rho(8) &= \varphi_1\rho(7) + \varphi_2\rho(6) = \frac{-577}{8000000}.
\end{aligned}$$

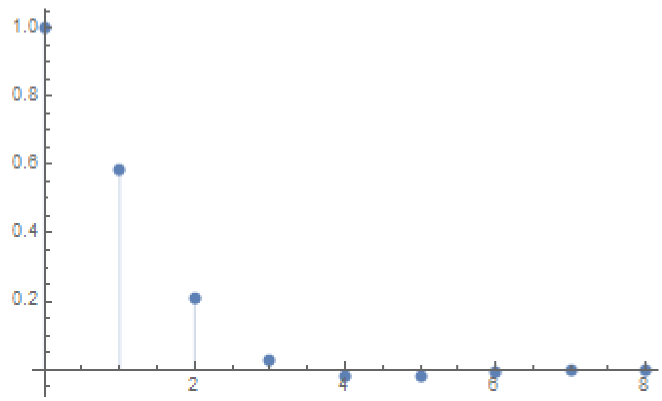
In general, for $k \geq 3$, we have $\rho(k) = \varphi_1\rho(k-1) + \varphi_2\rho(k-2)$ and for $k < 0$, $\rho(k) = \rho(-k)$.

(ii) Using formula (7.42) from Dufour (2008a), we have

$$\gamma(0) = \frac{\sigma^2}{1 - \varphi_1 \rho(1) - \varphi_2 \rho(2)} = \frac{30}{19}.$$

For general k, we can easily compute $\gamma(k) = \rho(k)\gamma(0)$ where $\rho(k)$ is given above.

(d) Plotting $\rho(k)$ for $k = 0, \ldots, 8$ yields



(e) We have

$$\begin{aligned}
\psi_0 &= 1; \\
\psi_1 &= \varphi_1 = \frac{7}{10}; \\
\psi_2 &= \varphi_1^2 + \varphi_2 = \frac{29}{100}; \\
\psi_3 &= \varphi_1 \psi_2 + \varphi_2 \psi_1 = \frac{63}{1000}; \\
\psi_4 &= \varphi_1 \psi_3 + \varphi_2 \psi_2 = \frac{-139}{10000}.
\end{aligned}$$

(f) The autocovariance function is

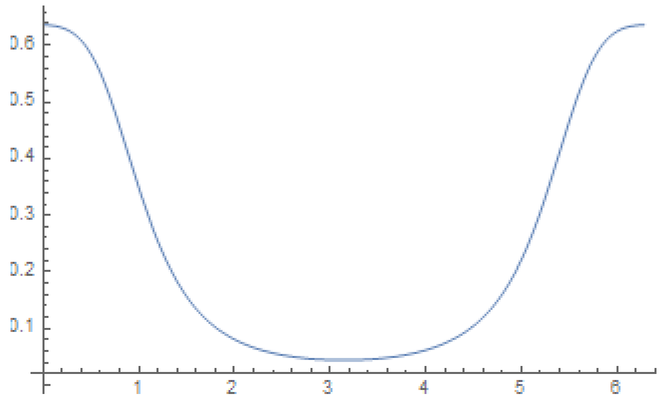
$$\gamma_x(z) = \sigma^2 \psi(z) \psi(z^{-1})$$

where $\psi(z) = \varphi(z)^{-1}$. In our particular case, the algebra simplifies to

$$\gamma_x(z) = \frac{100z^2}{(10 - 7z + 2z^2)(2 - 7z + 10z^2)}.$$

(g)

$$f_x(\omega) = \frac{\sigma^2}{2\pi[1 - \varphi_1 \exp(iw) - \varphi_2 \exp(2iw)][1 - \varphi_1 \exp(-iw) - \varphi_2 \exp(-2iw)]}.$$



(h)

$$\phi(1) = \frac{7}{12}, \quad \phi(2) = \frac{7}{10}, \quad \phi(3) = \phi(4) = 0.$$

The $MA(2)$ process $X_t = 10 + u_t - 0.75u_{t-1} + 0.125u_{t-2}$

Write this as

$$X_t = \mu + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, \quad \mu = 10, \quad \theta_1 = \frac{3}{4}, \quad \theta_2 = -\frac{1}{8}.$$

(a) Stationarity is automatic for all finite-order MA processes.

(b) This $MA(2)$ process is invertible because $\theta(z) = 1 - \theta_1 z - \theta_2 z^2$ has 2 roots 2 and 4 that both are outside the unit circle.

(c) We have:

(i) $E(X_t) = \mu = 10$,

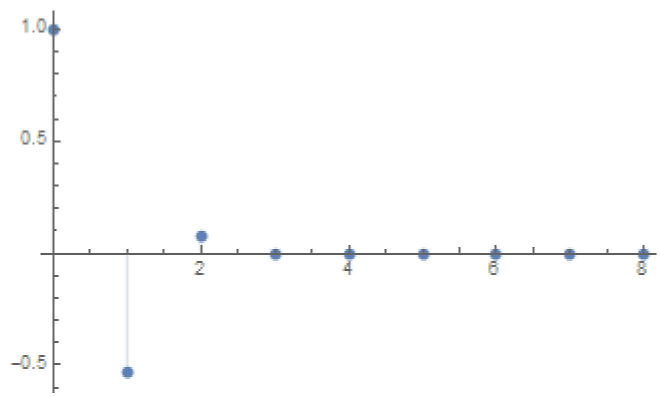
(ii) We have

$$\gamma(0) = \text{Var}(X_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2) = \frac{101}{64}$$
$$\gamma(1) = \sigma^2(-\theta_1 + \theta_1\theta_2) = \frac{-27}{32},$$
$$\gamma(2) = \sigma^2(-\theta_2) = \frac{1}{8},$$
$$\gamma(3) = \gamma(4) = \cdots = \gamma(8) = 0.$$

(iii) It follows that

$$\rho(0) = 1;$$
$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = -\frac{54}{101},$$
$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{8}{101},$$
$$\rho(3) = \rho(4) = \cdots = \rho(8) = 0.$$

(d) Plotting $\rho(k)$ for $k = 0, \dots, 8$ yields



(e) We have

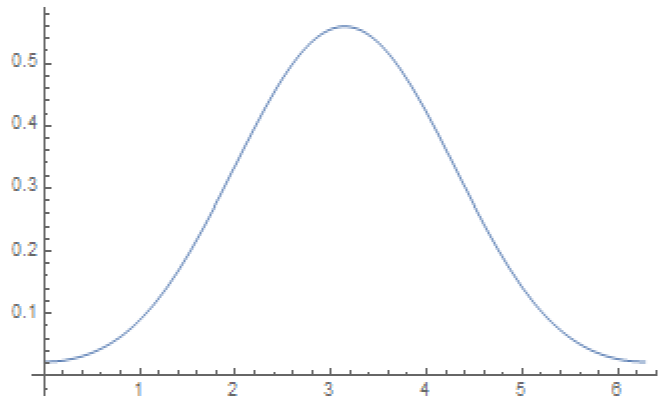
$$\psi_0 = 1,$$
$$\psi_1 = -\theta_1 = -\frac{3}{4},$$
$$\psi_2 = -\theta_2 = \frac{1}{8},$$
$$\psi_3 = 0,$$
$$\psi_4 = 0.$$

(f) The autocovariance generating function is

$$\begin{aligned}\gamma_x(z) &= \sigma^2\psi(z)\psi(1/z) \\ &= \sigma^2(1 - \theta_1 z - \theta_2 z^2)(1 - \theta_1/z - \theta_2/z^2) \\ &= \frac{(8 - 6z + z^2)(1 - 6z + 8z^2)}{64z^2}.\end{aligned}$$

(g) The spectral density is

$$f_x(\omega) = \frac{\sigma^2}{2\pi}\psi(e^{i\omega})\psi(e^{-i\omega})$$
$$= \frac{101 - 108\cos(\omega) + 16\cos(2\omega)}{128\pi}.$$



(h) We have

$$\phi(1) = -\frac{54}{101}, \quad \phi(2) = -\frac{68}{235}, \quad \phi(3) = -\frac{792}{5177}, \quad \phi(4) = -\frac{208}{2631}.$$

The $ARMA(1,1)$ **process** $X_t = 0.5X_{t-1} + u_t - 0.25u_{t-1}$

Write this as

$$(1 - \varphi_1 B)X_t = \bar{u} + (1 - \theta_1 B)u_t$$

where $\bar{u} = 0$, $\varphi_1 = 0.5$, $\theta_1 = 0.25$ and $u_t \sim N(0, \sigma^2)$ with $\sigma^2 = 1$.

(a) Stationary: yes because $1 - \varphi_1 z$ has a single root outside the unit circle.

(b) Invertible: yes because $1 - \theta_1 z$ has a single root outside the unit circle.

(c) We have:

(i) $E(X_t) = \frac{\bar{\mu}}{1-\varphi_1} = 0$,

(ii) We use formulas (8.39)–(8.41) from Dufour (2008a):

$$\gamma(0) = (1 - 2\varphi_1\theta_1 + \theta_1^2)\frac{\sigma^2}{1 - \varphi_1^2} = \frac{13}{12},$$
$$\gamma(1) = (1 - \theta_1\varphi_1)(\varphi_1 - \theta_1)\frac{\sigma^2}{1 - \varphi_1^2} = \frac{7}{24},$$

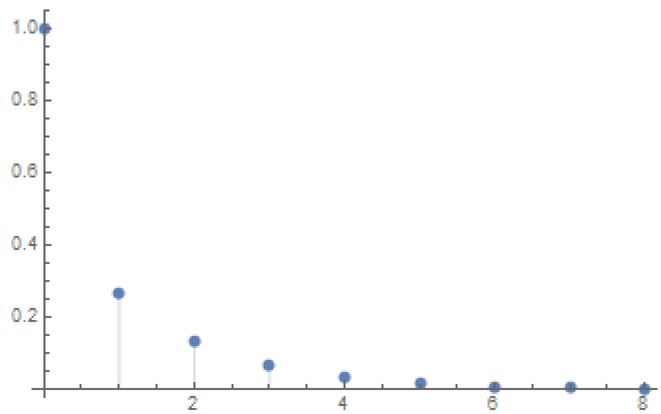
and $\gamma(k) = \varphi_1\gamma(k-1) = \varphi_1^{k-1}\gamma(1)$ for $k \geq 2$.

(iii) We have

$$\rho(0) = 1,$$
$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{1 - 2\varphi_1\theta_1 + \theta_1^2}{(1-\theta_1\varphi_1)(\varphi_1 - \theta_1)} = \frac{7}{26}$$

and $\rho(k) = \varphi_1 \rho(k-1) = \varphi_1^{k-1}\rho(1)$ for $k \geq 2$.

(d) Plotting $\rho(0), \ldots, \rho(8)$ yields



(e) We have

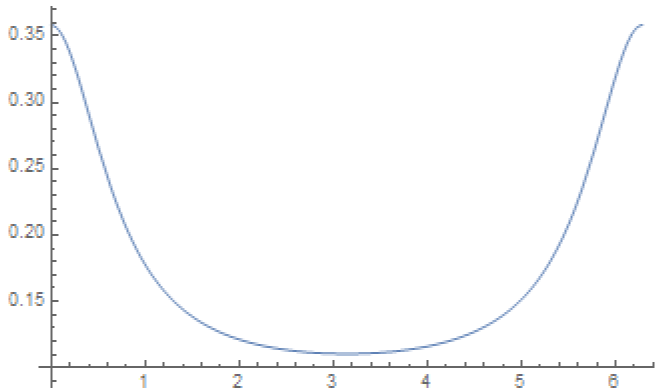
$$\psi_0 = 1,$$
$$\psi_1 = \varphi_1 - \theta_1 = \frac{1}{4}$$
$$\psi_2 = \varphi_1\psi_1 = \frac{1}{8}$$
$$\psi_3 = \varphi_1\psi_2 = \frac{1}{16}$$
$$\psi_4 = \varphi_1\psi_3 = \frac{1}{32}.$$

(f)

$$\gamma_x(z) = \sigma^2 \frac{\theta(z)\theta(z^{-1})}{\varphi(z)\varphi(z^{-1})} = \frac{4 - 17z + 4z^2}{8 - 20z + 8z^2}.$$

(g)

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{\theta[\exp(i\omega)]\theta[\exp(-i\omega)]}{\varphi[\exp(i\omega)]\varphi[\exp(-i\omega)]} = \frac{17 - 8\cos(\omega)}{2\pi(20 - 16\cos(\omega))}.$$

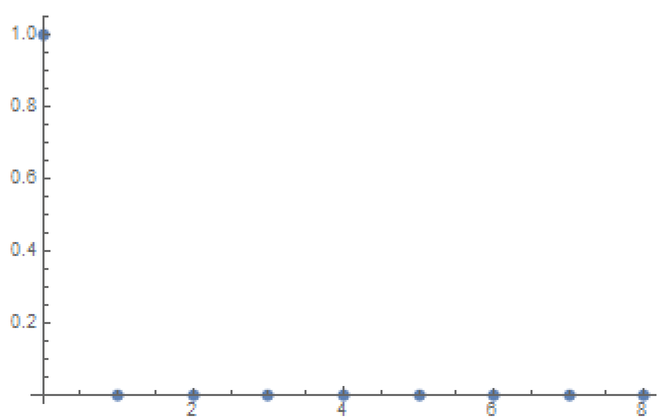


(h) Straightforward computation yields:

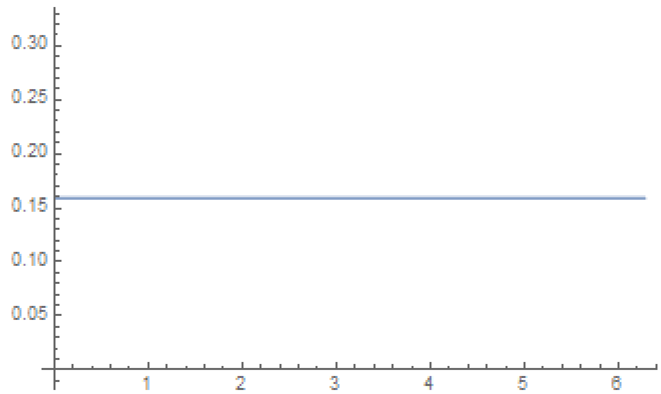
$$\phi(1) = \frac{7}{26},$$
$$\phi(2) = \frac{14}{209},$$
$$\phi(3) = \frac{56}{3345},$$
$$\phi(4) = \frac{224}{53521}.$$

The $ARMA(1,1)$ **process** $X_t = 0.5X_{t-1} + u_t - 0.5u_{t-1}$

This is the white noise process in disguise. So it is stationary and invertible. $\gamma(0) = 1$ and $\gamma(k) = 0$ for $k \neq 0$. Similarly, $\rho(0) = 1$ and $\rho(k) = 0$ for $k \neq 0$. Plotting $\rho(0), \ldots, \rho(8)$ is trivial:



We have $\psi_0 = 1$ and $\psi_k = 0$ for $k \geq 1$. The autocovariance generating function is just $\gamma_x(z) = 1$ whereas the spectral density is the constant $f_x(\omega) = \frac{1}{2\pi}$. Plotting the latter is trivial as well:



Finally, $\phi(1) = \cdots = \phi(4) = 0$ because the white noise can be seen as an $AR(0)$ process.

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